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THE AXIOMATIC STRUCTURE OF KNOWLEDGE AND PERCEPTION

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Abstract

Standard models of knowledge treat either information partitions or knowledge operators as primitives. The present paper starts with a new and arguably more intuitive primitive which is a binary relation (here called the imperception relation) on the states space. This relation expresses an individual's inability to distinguish between pairs of social states. Equivalences between axioms on this binary relation and the standard axioms for knowledge operators are established. Theorems concerning common knowledge and the convergence of posterior probabilities are worked out in the new framework. It is shown that Aumann's "agreeing-to-disagree" theorem remains valid even if we dispense with the axioms of Knowledge, Transparency and Wisdom, as long as the imperception relation satisfies a property called triangularity.

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Introduction

This paper is concerned with the foundations of knowledge and perception as used in economics, especially game theory. As such, its concern is with the internal consistency of knowledge. Given that a sentient being knows certain things, can we deduce that it knows something else? If I know something, do I know that I know that thing? These questions are very different from the questions with which the philosopher of knowledge is preoccupied. A lot of them deal with the meaning of knowledge and the relation between reality and knowledge. Descartes' famous remark, "I think, therefore I am", is a classic relational observation (despite Ogden Nash's effort to weaken it by noting that most people do not think but, nevertheless, "they are".)

Though the game theorist's agenda is probably less profound, it is technically exacting and involves long chains of hard logic. One of its major motivations is to locate the profusion of paradoxical results concerning rational behaviour in the implicit assumption of knowledge - for instance, the assumption that rationality is common knowledge - which so much of game theory takes for granted[1]. While this agenda remains wide open, our understanding of the algebra of knowledge has increased rapidly.

In the existing literature there are two alternative premises from which this algebra has been built up[2]. The first is to begin with the information partitions of the individuals and then to derive what they know. The second approach treats the knowledge operators as the primitives, imposes axioms on these operators and then derives the information partitions and defines common knowledge, truisms, and other useful terms. The aim of the present paper is to start from yet another primitive, the 'imperception relation'. Roughly speaking, if two social states, x and y, are related by individual i's imperception relation then i does not perceive any difference between these two states, or she cannot tell if x has occurred or y has occurred.

It is easy to see that by imposing a sufficient number of axioms on a person's imperception relation we can build up a system which is equivalent to the standard model based on the information partition or the knowledge operator. The advantage of this new primitive is that it enables us to examine what happens when all these axioms are not true. In addition, it allows us to evaluate the standard axioms of knowledge in a new light. Consider for instance, the axiom of Transparency, which claims that if a person knows something then she knows that she knows that thing. This is not the most transparent of axioms. Many people find it difficult to decide whether it is an appealing axiom or not. Hence, once we manage to establish equivalences between the axioms of knowledge and the axioms of imperception, we are better able to evaluate the axioms. The aim of section 2 is precisely to uncover the axiomatic structure of the new model and to chart its links with the traditional axioms.

After that the paper proceeds to examine some standard results, which were derived for a model where a player's knowledge is based on her information parition, in contexts where such information partitions do not exist. Section 3 discusses the concept of common knowledge and develops an alternative but equivalent definition. Section 4 establishes a theorem on the convergence of posterior probabilities in a model where individual knowledge operators fail to satisfy the standard axioms. We know from the work of Samet (1990) and Shin (1993) about the redundancy of the axiom of Wisdom. Samet (1990), in particular, has shown that Aumann's (1976) "agreeing to disagree" theorem remains valid as long as the axioms of Knowledge and Transparency remain valid. In section 4, it is proved that even Knowledge and Transparency are dispensible as long as the imperception relation satisfies a property called triangularity.

2. A Model of Perception

In the standard model of knowledge, for instance, the epistemic model (see Bacharach, 1985), every individual is endowed with a knowledge operator. To facilitate moving away from the standard model to the model of perception we may begin by briefly recapitulating the standard model of knowledge.

Let Ω be the set of all possible states of the world. It will be assumed that Ω is finite. Throughout this paper Ω is treated as fixed. Since we shall not delve into interpersonal matters till much later in the paper and certainly not in this section, let us focus attention here on a single individual. Her <u>knowledge</u>

operator, K, is a mapping:

$$K : 2^{\Omega} \rightarrow 2^{\Omega}$$

The interpretation of K is as follows. For every event $E \subset \Omega$, $K(E)$ is the event "she knows E". Five axioms, which have been often imposed on an individual's knowledge operator, are as follows[3].

[KO] $\quad K(\Omega) = \Omega$

[K1] $\quad \forall \; E, F \subset \Omega, \; K(E \cap F) = K(E) \cap K(F)$

[K2] $\quad \forall \; E \subset \Omega, \; K(E) \subset E$ (Axiom of Knowledge)

[K3] $\quad \forall \; E \subset \Omega, \; K(E) \subset K(K(E))$ (Axiom of Transparency)

[K4] $\quad \forall \; E \subset \Omega, \; \Omega \backslash K(E) \subset K(\Omega \backslash K(E))$ (Axiom of Wisdom)

These are not all independent axioms. As can be checked, K1-K4 imply KO (Bacharach, 1985, Proposition 1). However, K1, K2, K3 and K4 are independent of one another. It is well-known that if an individual's knowledge operator satisfies axioms K1-K4 then we have a model which is equivalent to Aumann's (1976) model of knowledge, where the primitive is an information partition and the knowledge operator is <u>derived</u> from the information partition.

A lot of effort has gone into assessing the strength and weakness of the five axioms, KO-K4 (see, eg., Binmore and Brandenburger, 1990). Though the knowledge operator is a more intuitive concept than an information partition, there is often a feeling that the knowledge operator, K, is not intuitive enough. If, for instance, a person is asked, "Whenever you know an event, do you know that you know the event?", most people will have difficulty answering the question because of an inability to fully comprehend it. Hence, it would be futile to check the appeal of the axiom of Transparency by asking people questions like the above one.

The aim of this paper, therefore, is to introduce yet another primitive in place of the knowledge operator or the information partition. In particular, we introduce the concept of an individual's '<u>imperception' relation</u>, T. If x, y $\in \Omega$, and xTy then this means that the individual cannot perceive any difference between x and y at y. If y occurs she will not know if x has occurred or y has occurred. Once we write down a model of knowledge based on the imperception relation, T, and <u>derive</u> the knowledge operator and axioms K1-K4 from axioms imposed on T, it becomes easier to assess the appeal of each of axioms K1 to K4. It turns out, for instance, that the axiom of Transparency, which is normally treated as self-evident, is a very strong assumption.

An individual's imperception relation, T, is a <u>binary relation</u> on Ω. That is,

$$T \subset \Omega \times \Omega$$

If (x,y) $\in$ T, we shall, at times, write this as xTy, and take it to mean that at state y, the individual cannot distinguish between x and y. Suppose that the only distinction between x and y is that, at y, Socrates has a stomache ache. Then, given that Socrates' pain usually had no behavioral manifestation, we could say that (x,y) $\in$ T where T is Xanthippe's imperception relation. That is, even when Socrates had a stomache ache, Xanthippe would not know it.

The present paper introduces the binary relation, T, not to shed light on Socrates and Xanthippe's marriage - though the above observation may provide an important clue to their marital discord - but because the axioms that can be imposed on T are more familiar and, therefore, more transparent than K0-K4. The three axioms that we shall have occasion to consider are:

[T1] $\forall x \in \Omega, (x,x) \in T$ (Reflexivity)

[T2] $\forall x,y \in \Omega, (x,y) \in T \rightarrow (y,x) \in T$ (Symmetry)

[T3] $\forall x,y,z \in \Omega, [(x,y) \in T, (y,z) \in T] \rightarrow (x,z) \in T$ (Transitivity)

Axiom T1 asserts that a person does not perceive any difference between x and x, for every state x. The axiom of symmetry requires that if the individual cannot tell the difference between x and y <u>at y</u>, then she cannot tell the difference between y and x <u>at x</u>. The "at y" and "at x" are emphasized because without these positional qualifiers, symmetry would be an unassailable axiom. In the information-partition approach to knowledge it is built into the framework that whether a person can tell the difference between x and y does not depend on where she is or which state actually occurs. It is <u>by assumption</u>, to rob Thomas Nagel (1986) of a book title, a "view from nowhere". But since our aim here is to examine the axioms closely it is important to remember that xTy means that the individual cannot tell if x has occurred or y has occurred, when y occurs[4]. Finally, T3 requires that if the individual concerned perceives no difference between x and y at y and y and z at z, then at z she must not be able to tell whether x has occurred or z has occurred.

Given the person's imperception relation it is possible to <u>derive</u> a knowledge operator. This may be done as follows. For all $x \in \Omega$, define

$$\overline{T}(x) := \{y \in \Omega \mid (y,x) \in T\}$$

Given T, we define K^T as the individual's derived knowledge operator if $K^T : 2^\Omega \to 2^\Omega$ such that, for all $E \subset \Omega$,

$$K^T(E) := \{x \in \Omega \mid \overline{T}(x) \subset E\}$$

The spirit of this definition is easy to appreciate. Suppose x is such that $\overline{T}(x) \subset E$. Then, though when x occurs, she will not know exactly which state has occurred (excepting in the special case where $\overline{T}(x)$ is a singleton), she will know that whatever has occurred is a part of E (since $\overline{T}(x) \subset E$). So she will know that E has occurred.

Reversing the above argument, if we are given a knowledge operator,we can <u>derive</u> the imperception relation. Given K, $T \subset \Omega \times \Omega$ will be called a <u>derived imperception relation</u> and denoted by T^K if the knowledge operator derived from T happens to be K. Given K, does T^K always exist and when it does exist is it unique? The next two lemmas answer these questions. It is shown that K can be derived from a T (that is, T^K exists) if and only if K satisfies axioms K0 and K1. It is shown that T^K is unique whenever it exists.

Lemma 1 Given a knowledge operator K, there exists an imperception relation T^K if and only if K satisfies axioms K0 and K1.

Proof Let T be an arbitrary imperception relation. That K^T must satisfy axiom K0 is obvious.

Let E, F $\in 2^\Omega$

$x \in K^T(E \cap F) \leftrightarrow \overline{T}(x) \subset E \cap F$

$\leftrightarrow \overline{T}(x) \subset E \ \& \ \overline{T}(x) \subset F$

$\leftrightarrow x \in K^T(E) \cap K^T(F)$

Hence K1 must be true. Hence if K is such that T^K exists, K must satisfy axiom K0 and K1.

To prove the reverse implication assume that K is a knowledge operator satisfying axioms K0 and K1. For all $x \in \Omega$, define $k(x)$ to be the smallest set, F, such that $x \in K(F)$.

To see that $k(x)$ exists for all x, define

$$C(x) := \{X \subset \Omega | x \in K(X)\}$$

By axiom K0, $\Omega \in C(x)$. By axiom K1, $k(x)$ is the intersection of all elements of $C(x)$, and hence it must exist, since Ω is finite.

Define an imperception relation T such that for all $x,y \in \Omega$, $(x,y) \in T \leftrightarrow x \in k(y)$. It follows that $T(x) = k(x)$, for all $x \in \Omega$.

Finally, I prove that, for all $E \subset \Omega$,

$K(E) = \{x \in \Omega | T(x) \subset E\}$

which would imply that $T = T^K$.

To prove this, consider $x \in K(E)$.

$\rightarrow$ $k(x) \subset E$, by the definition of T.

Next suppose x is such that $\overline{T}(x) \subset E$.

$\rightarrow$ $k(x) \subset E$.

$\rightarrow$ $x \in K(k(x))$, by definition of $k(x)$.

$\rightarrow$ $x \in K(k(x) \cap E)$, since $k(x) \subset E$.

$\rightarrow$ $x \in K(k(x)) \cap K(E)$, by axiom K1

$\rightarrow$ $x \in K(E)$. [Q.E.D.]

<u>Lemma 2</u> For every K, there is at most one derived imperception relation T^K.

<u>Proof</u> Suppose T and T′ are two distinct imperception relations. Hence, without loss of generality, there exists $x \in \Omega$ such that $\overline{T}'(x) \not\subset \overline{T}(x)$. Therefore, if $\overline{T}(x) := E$, then $K^T(E) \neq K^{T'}(E)$, since $x \in K^T(E)$ and $x \notin K^{T'}(E)$. [Q.E.D.]

It is easy to see that a knowledge operator satisfying K1-K4 is equivalent to an imperception relation satisfying T1-T3. This is stated formally in Theorem 1.

<u>Theorem 1</u> A knowledge operator, K, satisfies axioms K1-K4 if and only if there exists an imperception relation, T, satisfying axioms T1-T3 such that $K = K^T$.

The proof of this theorem is virtually obvious since a T satisfying T1-T3 is an equivalence relation, which generates an information partition on Ω, and a knowledge operator satisfying K1-

K4 is equivalent to a knowledge system defined from an information partition (Bacharach, 1985).

The axiom of Knowledge has been scrutinized and criticized in the literature. Binmore and Brandenburger (1990, p.119), for instance, have constructed arguments to "direct suspicion" at the "fundamental" axiom of Knowledge. The model developed here allows us to evaluate this axiom from more basic intuitions. As theorem 2 below shows, the Knowledge axiom is equivalent to the reflexivity property (axiom T1). Given that reflexivity is an eminently reasonable axiom, my model may be viewed as an argument in defence of the axiom of Knowledge.

<u>Theorem 2</u> T is reflexive if and only if K^T satisfies the axiom of Knowledge.

<u>**Proof**</u> Let T satisfy reflexivity and let $E \subset \Omega$. Suppose $y \in K^T(E)$. Hence $\overline{T}(y) \subset E$. Since T is reflexive, it follows $y \in E$. Hence K^T satisfies the axiom of Knowledge.

Next assume that T violates reflexivity. Hence for some $x \in \Omega$, $x \notin \overline{T}(x)$. Since, by definition, $x \in K^T(\overline{T}(x))$, it follows that $K^T(\overline{T}(x)) \not\subset \overline{T}(x)$. Hence K^T violates the axiom of Knowledge. [Q.E.D.]

Let me now turn to an evaluation of the axiom of Transparency. Whereas the axiom of Wisdom is widely criticized and, as remarked above, even the axiom of Knowledge has been questioned in the literature, the axiom of Transparency has gone largely unchallenged. It will now be shown that Transparency of the knowledge operator is equivalent to the transitivity of the imperception relation. It will then be argued that transitivity is perhaps the most suspect of the three assumptions, T1-T3. Hence, the axiom of Transparency should also be treated as vulnerable to criticism. Alfred Ayer (1974) would concur since he writes (on p.16) : "It can, indeed, be said of someone who hesitates, or makes a mistake, that he really knows what he is showing himself to be unsure of, the implication being that he ought, or is in a position, to be sure." Having expressed skepticism for the axiom of Transparency, Ayer proceeds to defend the axiom of Knowledge[5].

Theorem 3 **T is transitive if and only if K^T satisfies the axiom of Transparency.**

Proof Suppose T is transitive and $x \in K^T(E)$ for some E. Hence $\overline{T}(x) \subset E$.

Let $y \in \overline{T}(x)$ and $z \in \overline{T}(y)$. By the transitivity of T, $z \in \overline{T}(x)$.

Hence, $\overline{T}(y) \subset \overline{T}(x)$, for all $y \in \overline{T}(x)$

Therefore, $\overline{T}(x) \subset K^T(E)$, which, in turn, implies $x \in K^T(K^T(E))$, thereby establishing Transparency.

Next assume $K^T(E) \subset K^T(K^T(E))$, for all $E \subset \Omega$.

$\rightarrow [x \in K^T(E) \rightarrow \overline{T}(x) \subset K^T(E)]$

$\rightarrow [x \in K^T(E)\ \&\ (y,x) \in T \rightarrow y \in K^T(E)]$

$\rightarrow [x \in K^T(E),\ (y,x) \in T\ \&\ (z,y) \in T \rightarrow z \in E]$ (i)

Assume $(z,y) \in T$ & $(y,x) \in T$. Since, by definition, $x \in K^T(\overline{T}(x))$, it follows by (i) that

$$z \in \overline{T}(x)$$

or $(z,x) \in T$, which establishes transitivity. [Q.E.D.]

In the light of Theorem 3 we can evaluate the axiom of Transparency by evaluating the appeal of transitivity. As has been known for a long time the assumption of transitivity is tenuous precisely in the context of perception (Armstrong, 1951; Majumdar, 1958; Fishburn, 1970). Taking an example along the lines of Armstrong, suppose we are considering a sequence of states with varying numbers of grains of sugar in your cup of coffee. Hence, $\Omega = \{0,1,2,\ldots\}$. If state $2 \in \Omega$ occurs, it means that there are two grains of sugar in your coffee. It seems perfectly reasonable to assume that (i) you cannot perceive the difference between t and $t+1$, for all integers t, but (ii) you can tell the difference between t and $t+k$ where k is a large number. Hence your imperception relation violates the transitivity axiom. By Theorem 3 we must therefore reject the axiom of Transparency.

Before moving on, it is interesting as a digression to remark on a paradoxical-looking off-shoot of the above example. Let us for simplicity assume that (ii) is true for all $k \geq 2$. Now suppose there are 10 grains of sugar in your coffee, that is, $10 \in \Omega$ has occurred. From (i) and (ii) it follows that you will simply know that either 9 or 10 or 11 has occurred. Hence, at 10 the smallest event that you will know is $\{9,10,11\} \subset \Omega$.

This seems reasonable enough. However, note that 10 is the only state where you will think that 9,10 or 11 has occurred.

Afterall, if 9 had occurred, you would think that 8,9 and 10 has occurred. Hence from the fact that you know that one of 9,10 and 11 has occurred you should be able to "deduce" that 10 has occurred!

Fortunately, the above argument is not a matter of pure deduction. It presumes the validity of the axiom of Transparency. The fact that for most of us (i) and (ii) would be valid and most of us cannot, after sipping coffee, use our perception and "deduction" to say exactly how many grains of sugar there are in the coffee suggests to me that the axiom of Transparency is unacceptable[6].

Reflexivity having turned out equivalent to Knowledge, transitivity to Transparency, one is tempted to hazard equivalence for symmetry and Wisdom. But as the following example shows, that is not the case.

<u>Example 1</u> This example demonstrates that it is possible for K to satisfy the axiom of Wisdom but for T^K to violate symmetry.

Suppose $\Omega = \{x,y\}$, and

$K(\{x\}) = \{x,y\}$; $K(\{y\}) = \phi$; $K(\{x,y\}) = \{x,y\}$; $K(\phi) = \phi$.

Check that K satisfies the axiom of Wisdom. It is easy to compute the derived imperception relation:

$$T^K = \{(x,y), (x,x)\}$$

Since $(x,y) \in T^K$ and $(y,x) \notin T^K$, T^K violates symmetry. ∎

Observe in Example 1, K violates the Knowledge axiom since

$(x,y) = K(\{x\}) \not\subset \{x\}$. If we confined attention to knowledge operators which satisfy Knowledge, such an example would no longer be possible, since, as seen in the next theorem, Knowledge and Wisdom ensure symmetry.

Theorem 4 **If K satisfies the axioms of Knowledge and Wisdom, T^K must be symmetric.**

Proof Suppose K satisfies Knowledge and Wisdom; and T^K is not symmetric.

$\rightarrow$ There exists $x,y \in \Omega$, such that $x \notin \bar{T}^K(y)$ & $y \in \bar{T}^K(x)$.

Now T^K is either reflexive or not reflexive. If it is not reflexive, we know by Theorem 2 that K violates Knowledge. So assume T^K is reflexive.

$\rightarrow x \notin K(\bar{T}^K(y))$

$\rightarrow x \in \Omega \backslash K(\bar{T}^K(y))$

Next note that $y \in K(\bar{T}^K(y))$ & $y \in \bar{T}^K(x)$.

$\rightarrow x \notin K(\Omega \backslash K(\bar{T}^K(y)))$.

$\rightarrow$ K violates Wisdom [Q.E.D.]

A symmetric claim to Theorem 4 would assert that reflexivity plus symmetry would ensure Wisdom. The example that follows shows that such a claim would be false.

Example 2 This example demonstrates that it is possible for T to satisfy reflexivity and symmetry but for K^T to violate the axiom of Wisdom.

Suppose $\Omega = \{x,y,z\}$ and

$T = \{(x,x),\ (y,y),\ (z,z),\ (x,y),\ (y,x),\ (y,z),\ (z,y)\}$.

It follows that $\overline{T}(x) = \{x,y\}$, $\overline{T}(y) = \Omega$, $\overline{T}(z) = \{y,z\}$.

Let $E := \{y,z\}$. Clearly, $K^T(E) = \{z\}$. Hence $\Omega\backslash K^T(E) = \{x,y\}$. Therefore, $K^T(\Omega\backslash K^T(E)) = K^T(\{x,y\}) = \{x\}$, thereby revealing that K^T violates the axiom of Wisdom.

If, however, symmetry is combined with transitivity, the axiom of Wisdom is automatically ensured.

<u>Theorem 5</u> If T is transitive and symmetric, K^T must satisfy the axiom of Wisdom.

<u>Proof</u> Assume T satisfies transitivity and symmetry, and $x \in \Omega\backslash K^T(E)$, for some $E \subset \Omega$ and $x \in \Omega$. Now suppose

$$x \notin K^T(\Omega\backslash K(E)) \qquad \text{(ii)}$$

This implies $\overline{T}(x) \cap K(E) \neq \phi$.

Let $y \in \overline{T}(x) \cap K(E)$.

This implies $\overline{T}(y) \subset E$, since $y \in K(E)$ and $x \in \overline{T}(y)$, since $Y \in \overline{T}(x)$ and T is symmetric. The latter implies $\overline{T}(x) \subset \overline{T}(y)$, since T is transitive. Therefore, $\overline{T}(x) \subset E$, since $\overline{T}(y) \subset E$.

Hence, $x \in K^T(E)$.

This is a contradiction, which implies (ii) must be false. Hence K^T must satisfy the axiom of Wisdom since x and E were arbitrarily chosen. [Q.E.D.]

If we treat the imperception relation, T, as the primitive and ask the question as to what will T be like for a person of perfect perception, the answer is simple. It implies that for all x, $(x) = \overline{T}(x)$. That is, no state is confused with any other state. On the other hand, consider the property that no state can be unambiguously identified when it occurs. If T satisfies this property we shall say that T satisfies 'cognateness'. Formally, T satisfies <u>cognateness</u> if for all $x \in \Omega$, there exists $y \in \Omega \backslash \{x\}$ such that $(y,x) \in T$.

It is arguable that cognateness is a reasonable assumption in a world where perceptions are never perfect. Suppose, for instance, that Ω is not a primitive but constructed as follows. There are N independent propositions that can be true(1) or false(0). Then Ω could be thought of as the following Cartesian product.

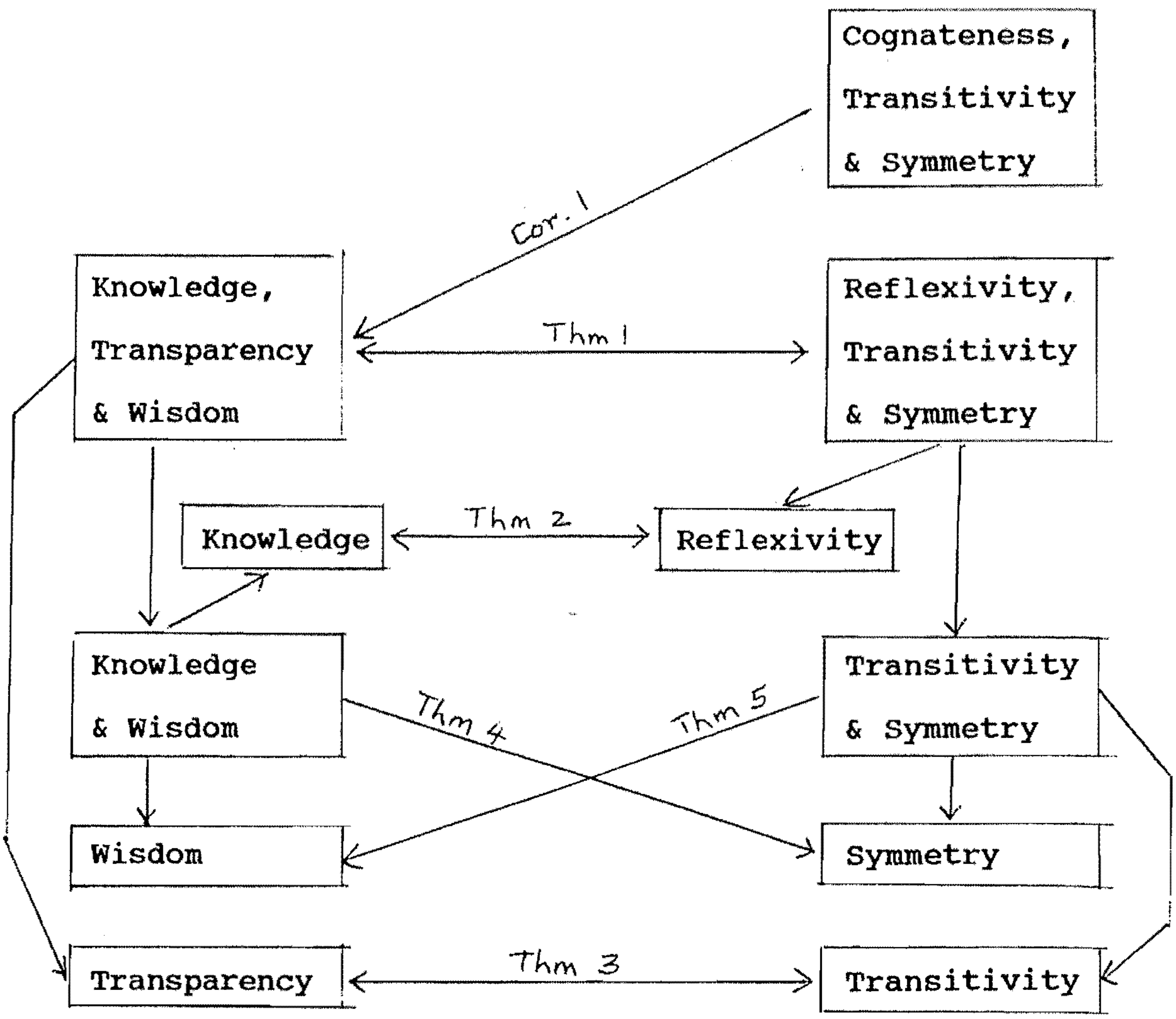
$\Omega = \{0,1\} \times \ldots \times \{0,1\}$ (N times).

Hence, a state of the world, x, is an N-tuple in which each element is 0 or 1. If the ith element in x is 1 then this means that in state x the ith proposition is true.

Now suppose there is one proposition (for instance, "It is now raining in Patagonia") the truth or falsity of which it is not possible for me to know. Clearly this is a reasonable assumption. But this immediately implies that my imperception relation satisfies cognateness. What is interesting is that in conjunction with other axioms cognateness has a lot of bite. It is easy to see that axioms T2 and T3 and cognateness imply axiom T1. This and Theorem 1 immediately imply the following:

Corollary 1 If T satisfies axioms T2 and T3 and cognateness, then K^T satisfies axioms K1-K4.

The central findings of this section are now summarised in the following implication diagram.



3. Common Knowledge

We are now ready to explore the algebra of interpersonal knowledge. From now on it will be assumed that I = {1,2,...,n} is the set of individuals. As before Ω is the finite set of all possible states. $(T_1, \ldots . T_n)$ is called a <u>model of perception</u> (MOP) if, for all $i \in I$, T_i is an imperception relation.

Given an MOP, $(T_1, \ldots T_n)$ let $(K_1, \ldots K_n)$ be an n-tuple of knowledge operators such that, for all i, K_i is the knowledge operator derived from T_i. In other words, $K_i = K_i^T$, for all $I \in I$. Given any $E \subset \Omega$, define

$$K(E) := K^1(E) := \bigcap_{i \in I} K_i(E)$$

Hence, K or K^1 denotes[7] the operator "everybody knows". Next, assuming that K^{t-1} is already defined, define

$$K^t(E) := \bigcap_{i \in I} K_i(K^{t-1}(E))$$

We shall follow the convention of writing $K^0(E) := E$. Given an MOP, an event E is <u>common knowledge</u> at state w if $w \in \bigcap_{t=1}^{\infty} K^t(E)$.

This is now a fairly standard definition of common knowledge and therefore I am not spending any effort motivating it or elaborating on it[8]. This definition of common knowledge is cumbersome to apply because it involves long recursive chains. One has to construct $K^1(E)$, $K^2(E)$,... and then take the intersection. It is for this reason that Aumann's definition of common knowledge based on the meet of information partitions is so useful. In what follows, yet another definition is developed which is based on the imperception relation. This definition has the additional advantage of being equivalent to Lewis' definition, above, even if the axioms K2-K4 are invalid.

Starting from the MOP, $(T_1, \ldots, T_n)$, define T to be the transitive closure[9] of $T_1 \cup \ldots \cup T_n$.

As before, let $T : \Omega \to 2^{\Omega}$ such that

$$\overline{T}(x) := \{y \in \Omega \mid (y,x) \in T\}$$

Theorem 6 **Given an MOP, an event E is common knowledge at state w if and only if $\overline{T}(w) \subset E$.**

Proof In brief, we have to prove:

$$x \in \bigcap_{t=1}^{\infty} K^t(E) \leftrightarrow \overline{T}(x) \subset E.$$

First note that:

$$\overline{T}(x) \subset K^t(E) \to x \in K^{t+1}(E) \qquad \text{(iii)}$$

To see this, observe that the left-hand side implies

$$\overline{T}_i(x) \subset K^t(E),\ \forall\ i,\ \text{since } \overline{T}_i(x) \subset \overline{T}(x)$$

$$\to x \in K_i(K^t(E)),\ \forall\ i.$$

$$\to x \in K^{t+1}(E).$$

Next we shall prove that

$$\overline{T}(x) \subset E \to \overline{T}(x) \subset K^t(E),\ \forall\ t. \qquad \text{(iv)}$$

Suppose $\overline{T}(x) \subset E$ and $\overline{T}(x) \subset K^{t-1}(E)$.

Let $y \in \overline{T}(x)$.

$$\to \overline{T}_i(y) \subset \overline{T}(x),\ \forall\ i,\ \text{since T is transitive}$$

$$\to \overline{T}_i(y) \subset K^{t-1}(E),\ \forall\ i.$$

$$\to y \in K_i(K^{t-1}(E)),\ \forall\ i.$$

$$\to y \in K^t(E).$$

Hence $\overline{T}(x) \subset K^t(E)$.

By induction, $\overline{T}(x) \subset K^t(E),\ \forall\ t.$

(iii) and (iv) imply $[\overline{T}(x) \subset E \to x \in K^{t+1}(E),\ \forall\ t = 0,1,.]$

Hence $\overline{T}(x) \subset E \to x \in \bigcap_{t=1}^{\infty} K^t(E)$.

In order to prove the other implication, suppose $x \in \bigcap_{t=1}^{\infty} K^t(E)$ and $y \in \overline{T}(x)$.

Hence, there exists an integer k, $z_1, \ldots, z_k \in \Omega$ and $i_1, \ldots, i_{k+1} \in I$ such that

$z_1 \in \overline{T}_i\ (x)$ (1)

$z_2 \in \overline{T}_i\ (z_1)$ (2)

and continuing in the same fashion...

$z_k \in \overline{T}_i\ (z_{k-1})$ (k)

and $y \in \overline{T}_i\ (z_k)$ (k+1)

Now, $x \in \cap K^t(E)$ implies

$x \in K^{k+1}(E)$

$\rightarrow z_1 \in K^k(E)$, by (1).

$\rightarrow z_2 \in K^{k-1}(E)$, by (2).

and proceeding by the same logic, we get

$z_k \in K^1(E)$, by (k).

$\rightarrow y \in E$, by (k+1).

This proves that

$x \in \bigcap_{t=1}^{\infty} K^t(E) \rightarrow T(x) \subset E.$ [Q.E.D.]

4. Common Posterior Probabilities

The aim of this section is to generalise Aumann's (1976) 'agreeing-to-disagree' theorem. Using a framework of knowledge where axioms K0-K4 were valid for each individual, Aumann showed that if the players have common priors and their posterior probabilities of some event are common knowledge, then the posteriors must be identical to one another. This theorem has been the provocation for a lot of related work[10]. In keeping with the motivation of this paper, my aim here is to explore whether such a theorem remains valid in the absence of axioms K0-K4 or, equivalently, axioms T1-T3. In particular, since we argued that the transitivity of the imperception relation may be an untenable assumption in many situations, it will be worthwhile asking, if we can get Aumann-type results, while eschewing axiom T3.

In addition to the finite set of states, Ω, and an MOP, $(T_1, \ldots T_n)$, we shall now assume that we are given a prior probability, p, which is common to all agents. Thus $p : \Omega \rightarrow [0,1]$ such that

$\sum_{x\in\Omega} p(x) = 1$

Given an event E and state x, person i's posterior probability of E is denoted by $p_i(E|x)$ and defined

$$p(E|x) := \frac{p(E\cap\overline{T}_i(x))}{p(\overline{T}_i(x))} := p(E|T_i(x))$$

where for $X \subset \Omega$, p(x) should be taken to be $\sum_{x\in\Omega} p(x)$.

My aim now is to establish that if the players' posteriors are common knowledge, they must be identical. It will be shown that this result does not require that each player satisfies axioms T1-T3. However, though each of these axioms can be dispensed with, we do need some restrictions on the imperception relations.

We shall say that an imperception relation, T_i, satisfies triangularity (or is triangular) if, for all $x,y,z \in \Omega$, $[(z,x) \in T_i$ & $(z,y) \in T_i] \rightarrow [(y,x) \in T_i]$.

It is easy to check that triangularity does not imply reflexivity, transitivity or symmetry; but that transitivity and symmetry imply triangularity. From Samet (1990) we know that the Aumann-type result remains valid in the absence of the axiom of Wisdom. Since Theorems 2 and 3 assure us of the equivalence of reflexivity and the axiom of Knowledge, and also of transitivity and the axiom of Transparency, the next theorem shows that the axioms of Knowledge and Transparency can also be dispensed with, while establishing an Aumann-type result.

Theorem 7 **Consider an MOP such that, for all i, T_i is triangular. If the posterior probabilities of an event E of the n players are given by $q_1,\ldots,q_n$ and this is common knowledge, then $q_1=q_2=\ldots=q_n$.**

Proof Suppose the MOP is such that T_i is triangular for all i. Let E_i be the event: "Person i's posterior probability of event E is q_i". Hence,

$$E_i := \{x \in \Omega \mid p_i(E \mid x) = q_i\}$$

Assume that state w has occurred and the events $E_1,\ldots,E_n$ are common knowledge. We have to show that $q_1 = q_2 = \ldots = q_n$.

Given Theorem 6 and the fact that $E_1\ldots E_n$ are common knowledge at w, we know that

$$\bar{T}(w) \subset E_1 \cap E_2 \cap\ldots\cap E_n \qquad \text{(v)}$$

Consider a person i. Note that triangularity implies that there exists a finite number of states $x_1,\ldots,x_m$ such that $(\overline{T}_i(x_1),\ldots,\overline{T}_i(x_m))$ is a disjoint collection of sets such that

$$\overline{T}(w) \subset \overline{T}_i(x_1)\cup\ldots\cup \overline{T}_i(x_m). \qquad \text{(vi)}$$

To see this first note an obvious implication of T_i being triangular.

$$\forall\ a,b \in \Omega,\ b \notin \overline{T}_i(a) \rightarrow \overline{T}_i(b)\cap\overline{T}_i(a) = \Phi$$

Now pick any element from $\overline{T}(w)$ and call it x_1. Next pick any element from $\overline{T}(w)\backslash\overline{T}_i(x)$ and call it x_2. Next pick an element from $\overline{T}(w)\backslash\overline{T}_i(x_1)\cup\overline{T}_i(x_2))$ and call it x_3. And so on, till the relative complement becomes empty. Since Ω is finite, this establishes the claim in (vi).

Hence,

$$p(E|\overline{T}(w)) = \frac{p(E\cap\overline{T}_i(x_1))+\ldots+p(E\cap\overline{T}_i(x_m))}{p(\overline{T}_i(x_1))+\ldots+p(\overline{T}_i(x_m))}$$

Since by (v), we know $\overline{T}(w) \subset E_i$, hence

$$[p(E\cap\overline{T}_i(x_j))]/[p(\overline{T}_i(x_j))] = p_i(E|x_j) = q_i,\ j = 1,\ldots,m.$$

Therefore,

$$p(E|\overline{T}(w)) = \frac{q_ip(\overline{T}_i(x_1))+\ldots+q_ip(\overline{T}_i(x_m))}{p(\overline{T}_i(x_1))+\ldots+p(\overline{T}_i(x_m))}$$

It follows that $q_1=\ldots=q_n$, since $p(E|T(w))$ is independent of i.

[Q.E.D.]

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FOOTNOTES

1. See, for instance, Luce and Raiffa (1957), Binmore (1987), Rosenthal (1987), Reny (1986), Basu (1990), Bonano (1991) and Brandenburger (1992).

2. See, for instance, Aumann (1976), Milgrom (1981), Bacharach (1985), Werlang (1986) and Binmore and Brandenburger (1990). For a discussion of the differences between the economist's approach and the logician's approach, see Bonano (1994).

3. For discussions of these axioms, see Bacharach (1985) and Binmore and Brandenburger (1990).

4. For a discussion of "positional" qualifiers and their role in a larger philosophical context, see Sen (1993).

5. And a more theatrical critique of the axioms of both Transparency and Wisdom occurs in Tom Stoppard's Jumpers. When Dotty asks, "Do you find it incredible that a man with a scientific background should be Archbishop of Canterbury?", George's reply betrays his rejection of Wisdom:

"How the hell do I know what I find incredible? Credibility is an expanding field...Shear disbelief hardly registers on the face before the head is nodding with all the wisdom of instant hindsight."

Later, George casts aside Transparency with the following refrain:

"How does one know what it is one believes when it is so difficult to know what it is one knows."

6. I am now persuaded that this 'paradoxical' example can be interpreted in different ways to reach different conclusions. I owe this to conversations with Jorgen Weibull. However, I remain convinced of what is a central message of this discussion, to wit, that the axiom of Transparency is often untenable and should be used with caution.

7. The K in this section is therefore not to be confused with K in the previous section, which was actually Ki for some fixed person i.

8. This definition is generally attributed to Lewis (1969). I have however argued elsewhere (Basu, 1994) that though this definition is in the spirit of Lewis, it is not a precise formulation of Lewis' definition.

9. Once again, T, in this section is not to be confused with T in the previous section, which was one fixed person's imperception relation.

10. See, for instance, Geanakoplos and Polemarchakis (1982), Milgrom and Stokey (1982), Samet (1990), Tirole (1991), Ferrante (1991), Shin (1993), Babu (1994). Bacharach (1985) and Parikh and Krasucki (1990) show that such convergence results can be extended from posterior probabilities to other general functions defined on the power set of Ω.

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