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Kaushik Basu

Delhi School of Economics

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Delhi School of Economics
Delhi 110 007 INDIA

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Kaushik Basu
Delhi School of Economics
Delhi 110007
India

Abstract

It is arguable that in many two-person bargaining situations disagreement leads to a set of possible payoffs with no probabilities attached to the elements of the set. Axioms are developed for bargaining games of this kind and solution concepts are derived from these axioms. Particular attention is paid to what are here called the 'max-max' and 'rectangular general' solutions. The latter can be applied to an important sub-class of bargaining games where the disagreement set is equal to the feasible set.

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BARGAINING WITH SET-VALUED DISAGREEMENT

Kaushik Basu

1 Introduction

The standard ingredients of a two-person bargaining problem are : a *feasible set*, S, which is a subset of $\mathbb{R}^2$ and a *disagreement point* (or a *threat point*), $d \in S$. The interpretation is that the two players are free to cooperate and choose any point $x \equiv (x_1, x_2) \in S$, where x_i denotes the amount of utility that player i gets; but if they fail to cooperate then the players get d, that is, player 1 gets d_i utils and 2 gets d_2 utils.

Is it reasonable to suppose that there will exist such a well-defined point like the disagreement point in all bargaining situations? In the early formulations, like that of Edgeworth (1991), where players were assumed to have initial endowments of goods and could fall back on these if their bargains failed, the answer to this question may have been a yes. But the Edgeworth formulation is not the only one. In the modern approach, based on Nash's (1950) agenda, the bargaining problem is often thought of as one deriving from a standard normal-form game. The feasible set S is then simply the feasible set of payoffs of the underlying normal-form game, G. In that case it is not obvious what the disagreement point is. A reasonable assumption is that if the cooperative bargain fails the players will play the game G non-cooperatively. In other words the "threat" is not a "point" but the "fact that the game will be played non-cooperatively".

If a game is played noncooperatively, it is not clear that both players will always have a unique expectation of payoffs for themselves. After all, the underlying game, G, may have several Nash equilibria. Moreover, if the players approach the game only with the common knowledge of their rationality, then they would expect to get payoffs from within the set of rationalizable strategies, but they would not be able to attach probabilities to the different elements of the set[1]. If there is some preplay communication (and this is to be expected since the process of negotiation can itself transmit information to one another)[2] we may expect players to choose strategies which are within a curb or a tight curb set (see Basu and Weibull, 1991)[3], but even in this case disagreement does not result in a point-valued or unique payoff.

In any case it seems reasonable to *allow* for the possibility that in the event of a breakdown in bargaining there may exist a *set* of disagreement points and it may not be possible to attach probabilities to the elements in the set. Hence, in this paper, bargaining games are described by two sets : S, the set of feasible payoffs, and T, the threat set. Solution concepts for such games are derived axiomatically. In selecting axioms concerning the responsiveness of the solution to changes in the threat set T, I draw on the literature on extending a binary relation over a set to a binary relation over the power set of that set (see, for example, Kannai and Peleg, 1984; Barbera and Pattanaik, 1984). This literature is used to extract axioms for the present context in Section 3. Some of the results presented in Section 3 may be of interest, independently of this paper, in the context of the literature on ranking sets of alternatives.

Section 2 introduces the notation and preliminary axioms. In Section 4 a solution for bargaining games called the max-max solution is defined and axiomatized. An alternative axiomatization for the max-max solution is derived in section 5.

One important kind of bargaining game is a (T, S) such that $T = S$. We call this a bargaining game with 'complete uncertainty'. In such a game, if the bargain fails, the players simply know that anything can happen. This would be reasonable if, for instance, in the underlying strategic form game the 'minimal tight curb set' is the set of all strategies. The max-max solution unfortunately cannot always be applied to much games where the threat set coincides with the feasible set. (The reason for this will be clear from Sections 2 and 3.) In Section 5, a solution concept called the rectangular general solution is introduced, and axiomatized. This solution gives us a way for splitting payoff between players in bargaining games with complete uncertainty.

2 Notation and Preliminary Axioms

Let Z be the collection of all non-empty compact subsets of $\mathbb{R}^2$. For all $S \in Z$, we define

$$\text{1-min } S := \min\{x_1 | (x_1, x_2) \in S, \text{ for some } x_2\}$$

$$\text{2-min } S := \min\{x_2 | (x_1, x_2) \in S, \text{ for some } x_1\}$$

1-max S and 2-max S are defined likewise.

If $S \in Z$, the *rectangular hull* of S, denoted by $H(S)$, is the smallest set containing S, which is a Cartesian product of two intervals. Thus

$$H(S) = [\text{1-min } S, \text{ 1-max } S] \times [\text{2-min } S, \text{ 2-max } S].$$

A *general bargaining game* is an ordered pair (T, S) such that $T, S \in Z$, S is convex and $T \subset S$. To start with, however, I shall focus on a slightly less general bargaining game, which will be referred to simply as a 'bargaining game'. A *bargaining game* is a general bargaining game, (T, S), such that $H(T) \subset S$. Let Ω be the collection of all bargaining games. A *solution* is a mapping

$f : \Omega \to \mathbb{R}^2$ such that, for all $(T, S) \in \Omega, f(T, S) = (f_1(T, S), f_2(T, S)) \in S$. An important subclass of Ω is

$$\bar{\Omega} = \{(T, S) \in \Omega | T \text{ is a singleton } \}.$$

A bargaining game that belongs to $\bar{\Omega}$ will be referred to as a ***traditional bargaining game*** since the Nash bargaining problem belongs to this class.

Here is some more notation which is used below. For all $x, y \in \mathbb{R}^2$,

$$[x \geq y] \leftrightarrow [x_i \geq y_i, \quad \text{for all} \quad i]$$

$$[x > y] \leftrightarrow [x \geq y, \quad \text{and not} \quad y \geq x]$$

$$[x \gg y] \leftrightarrow [x_i > y_i, \quad \text{for all} \quad i].$$

Now we impose axioms on f. The first axiom is trivial and is often not stated separately but is built into the description of bargaining games (see, e.g., Friedman, 1986, and Thompson and Lensberg, 1989). It is also similar in spirit to the axiom of "Independence of Non-individually Rational Points" used in the literature (Peters, 1986, Chun and Thomson, 1989). Note that the next five axioms restrict the behaviour of the solution, f, over only the domain $\bar{\Omega}$. Nevertheless in the end we shall have a characterization of f over Ω.

Axiom I (Irrelevance) : If $(\{d\}, S), (\{d\}, S') \in \bar{\Omega}$ are such that $[x \in S] \to [\exists\, x' \in S'$ such that $x' \geq x]$ and $S' \subset S$, then $f(\{d\}, S) = f(\{d\}, S')$.

The next three axioms are standard.

Axiom P (Pareto) : For all $(T, S) \in \bar{\Omega}$, there does not exist $y \in S$ such that $y > f(T, S)$.

Axiom S (Symmetry) : Define a function $h : Z \to Z$ such that, for all $S \in Z$, $h(S) = \{x \in \mathbb{R}^2 | (x_2, x_1) \in S\}$.
Then for all $(T, S) \in \bar{\Omega}$, $f(h(T), h(S)) = h(\{f(T, S)\})$.

Axiom A (Affine Invariance) : If $(\{d\}, S), (\{d'\}, S') \in \bar{\Omega}$ such that there exist $b_1, b_2 \in \mathbb{R}_+$ and $c_1, c_2 \in \mathbb{R}$ so that

$$d_i = c_i + b_i d_i', \quad i = 1, 2$$

and $\quad [x \in S] \leftrightarrow [\exists\, x' \in S' \text{ such that } x_i = c_i + b_i x_i', \quad i = 1, 2]$

then $\quad f_i(\{d\}, S) = c_i + b_i f_i(\{d'\}, S'), \quad i = 1, 2.$

The next axiom is that of Kalai and Smorodinsky (1975). It is possible to use, instead, Nash's axiom of 'Independence of Irrelevant Alternatives'. This is discussed in Section 4. Given that we are currently using the Kalai-Smorodinsky framework, it is possible for us to take advantage of this and to relax the convexity requirement of the feasible set in bargaining games, as in Anant, Basu and Mukherji (1990) or Conley and Wilkey (1989).

Axiom M (Monotonicity) : Let $(\{d\}, S), (\{d\}, S') \in \bar{\Omega}$ and $S \subset S'$. If

$$\text{i-max } S \cap \{x \in S | x \geq d\} = \text{i-max } S' \cap \{x \in S' | x \geq d\},$$

then $\quad f_j(\{d\}, S') \geq f_j(\{d\}, S),$

where $\quad j \in \{1, 2\} - \{i\}.$

Up to here everything is standard and makes no use of the fact that our disagreement 'point' is not a point at all but a set. In choosing axioms to characterize the effect of changes in the disagreement or threat set on the solution, I draw on the literature which considers the problem of deriving an agent's preference over sets of alternatives, given a prespecified preference over the alternatives (see, eg., Barbera, Barrett and Pattanaik, 1987). I use these ideas to derive preferences over threat sets for each player and then impose an axiom that if both agents find T and T' equally good then a change of the threat set from T to T' does not change the value of the solution. But before getting to that we need an interlude on ranking sets of alternatives.

3 Ranking Sets of Alternatives

This section may be read independently of the previous ones.

Let U be the universal set of alternatives and let V be the set of all non-empty subsets of U. Let R be an ordering over U and $\geq$ a quasi-ordering (i.e. a reflexive and transitive binary relation) over V. The next three properties capture the idea that $\geq$ is, in some sense, generated by R. In what follows the asymmetric and symmetric parts of R are denoted by P and I and the asymmetric and symmetric parts of $\geq$ are denoted by $>$ and $\sim$.

Property 1 : If $A, B \in V$ such that, for all $(x, y) \in A \times B$, xPy, then $A \geq A \cup B \geq B$ and $A > B$.

Property 2 : If $A, B, C \in V$ such that $A \geq B$ and $A \cap C = B \cap C = \phi$, then

$$A \cup C \geq B \cup C.$$

Property 3 : If $A, B \in V$ such that, for all $(x, y) \in A \times B, xIy$, then $A \sim B$.

Note that property 1 is a variant of the well-known 'Gardenfors Principle' (Gardenfors, 1976). Property 2 is an 'independence' axiom. Property 3 is virtually a property of reflexivity. The only reason one does not encounter it in the traditional literature is that in such literature the underlying relation R is usually treated as a linear order, which implies that property 3 is trivially valid.

If $\geq$ is a quasi-ordering satisfying properties 1-3 we say that $\geq$ is *generated by R*. For all $A \in V$, the sets of greatest and least elements of A are defined as follows:

$$g(A) := \{x \in A | xRy, \text{ for all } y \in A\}$$

$$\ell(A) := \{x \in A | yRx, \text{ for all } y \in A\}.$$

Lemma 1 : Let $\geq$ be a quasi-ordering on V generated by R. Let $A, B \in V$ be such that $g(A)$, $g(B)$, $\ell(A)$, $\ell(B)$ are non-empty and, for all $(x,y) \in g(A) \times g(B)$, xIy and, for all $(x,y) \in \ell(A) \times \ell(B)$, xIy. Then $A \sim B$.

Proof: Assume that the hypothesis of the lemma is true. Suppose first that $g(A) = \ell(A)$. Then it follows by an obvious application of property 3 that $A \sim B$.

Next suppose that, for some $(x,y) \in g(A) \times \ell(A)$, xPy. It will first be proved that $A \sim g(A) \cup \ell(A)$. If $A = g(A) \cup \ell(A)$, then $A \sim g(A) \cup \ell(A)$, by the reflexivity of $\geq$. Hence suppose that, there exists $Y \in V$, such that

$$A = g(A) \cup \ell(A) \cup Y.$$

Since, for all $(x,y) \in g(A) \times Y$, xPy, hence

$$g(A) \geq g(A) \cup Y, \text{ by property 1}$$

$$g(A) \cup \ell(A) \geq A, \text{ by property 2}$$

By a similar argument, $A \geq g(A) \cup \ell(A)$.

Hence $A \sim g(A) \cup \ell(A)$.

In brief, this establishes that, for all $X \in V$, with nonempty $g(X)$ and $\ell(X)$, $X \sim g(X) \cup \ell(X)$.

Next, observe that an immediate implication of property 2 is the following, which will be referred to as :

(2′) For all $A, B, C \in V$, $[A \sim B,\ A \cap C = B \cap C = \phi] \rightarrow [A \cup C \sim B \cup C]$.

By property 3, we have : $g(A) \sim g(B)$.

Hence, $g(A) \cup \ell(B) \sim g(B) \cup \ell(B)$, by (2′).

By property 3, we have : $\ell(A) \sim \ell(B)$.

Hence $g(A) \cup \ell(A) \sim g(A) \cup \ell(B)$, by (2′).

By the transitivity of $\geq$, $g(A) \cup \ell(A) \sim g(B) \cup \ell(B)$. The proof is completed by observing that $A \sim g(A) \cup \ell(A)$ and $B \sim g(B) \cup \ell(B)$. ■

Lemma 1 is related to a result of Arrow and Hurwicz (1972) which shows that in evaluating a collection $X \in V$ the only things that matter are the best and worst elements of X. In different settings this result has cropped up several time in the literature on ranking sets cited above.

Observe that Lemma 1 does not guarantee the existence of a quasi-ordering $\geq$ satisfying properties 1-3. It merely says that if such a quasi-ordering exists, then it must have some characteristics. Since, starting with the work of Kannai and Peleg (1984), the literature on generating ranking over sets has demonstrated that this area is riddled with non-existence problems, it is important to demonstrate existence.

Lemma 2 : There exists a quasi-ordering, $\geq$, on V, satisfying properties 1-3.

Proof : Define a binary relation $\textcircled{\geq}$ on V follows:

$[A \textcircled{\geq} B] \leftrightarrow$ [for all $x \in B$, there exists $y \in A$, such that yRx].

The reflexivity of $\textcircled{\geq}$ follows from the reflexivity of R. Assume $A \textcircled{\geq} B$ and $B \textcircled{\geq} C$. Then, for all $z \in C$, there exists $w \in B$, such that wRz. But $A \textcircled{\geq} B$ implies there exists $x \in A$ such that xRw. Since R is transitive, xRz. Hence $A \textcircled{\geq} C$. Thus $\textcircled{\geq}$ is a quasi-ordering.

Let $A, B \in V$ such that, for all $(x, y) \in A \times B$, xPy. Then, for all

$x \in A \cup B$, there exists $y \in A$ such that yRx. To see this note that if $x \in A$, then $y = x$ does the job; and if $x \in B$, and $y \in A$ would do. Hence $A \textcircled{\geq} A \cup B$. It is easy to see that $A \cup B \textcircled{\geq} B$. Hence $\textcircled{\geq}$ satisfies property 1.

Suppose $A \textcircled{\geq} B$ and $A \cap C = B \cap C = \phi$. Let $x \in B \cup C$. Then, either $x \in B$ or $x \in C$. If $x \in C$ then there exists $y \in A \cup C$ such that yRx. To see this choose $y = x$.

If $x \in B$ then there exists $y \in A$ (and therefore $y \in A \cup C$) such that yRx, since $A \textcircled{\geq} B$.

This establishes property 2.

Property 3 is obvious. ■

Remark 1 : There are other binary relations satisfying reflexivity, transitivity and properties 1-3. The reader may check that this is true for the following binary relation $\geq *$:

$$[A \geq * B] \leftrightarrow [\forall\, x \in A, \exists\, y \in B \text{ such that } xRy].$$

Remark 2 : Since $\textcircled{\geq}$, defined above, is complete what the proof establishes is more than lemma 2. It shows that there exists on *ordering*, $\geq$, satisfying properties 1-3.

Before moving on it is worthwhile taking note of the pitfalls of non-existence in this area. Small strengthenings of the conditions on $\geq$ quickly result in non-existence. Consider, for instance, a variant of property 2.

Property 2* : If $A, B, C \in V$ such that $A > B$ and $A \cap C = B \cap C = \phi$, then $A \cup C > B \cup C$.

2* looks like a very reasonable condition but the next lemma shows how it can tip the balance.

Lemma 3 : If U consists of at least four strictly ranked elements (in terms of R), then there is no quasi-ordering $\geq$ satisfying properties 1, 2, 2* and 3.

Proof : Assume $\geq$ is a quasi-ordering satisfying properties 1, 2, 2' and 3 and that $x, y, z, w \in U$ and xPy, yPz, zPw. By property 1, $\{y\} > \{z\}$. By property 2*, $\{x, y, w\} > \{x, z, w\}$. But since $g(\{x, y, w\}) = g(\{x, z, w\}) = x$ and $\ell(\{x, y, w\}) = \ell(\{x, z, w\}) = z$, and we know from lemma 1 that all $\geq$ satisfying properties 1,2 and 3, would declare such states as equally good, we have $\{x, y, w\} \sim \{x, z, w\}$. This contradiction establishes the non-existence of $\geq$. ■

We now proceed to use the ideas of these sections to derive axioms concerning the effect of changes in threat sets in our bargaining problem.

4 An Axiomatization of the 'Max-max' Solution

For all $i \in \{1, 2\}$, let R_i be the following binary relation on $\mathbb{R}^2$. For all $x, y \in \mathbb{R}^2, xR_iy \leftrightarrow x_i \geq y_i$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The next axiom asserts that if the threat set changes from T to T' but both players find T and T' equally desirable, then the value of the solution must not change.

Axiom T (Threat Invariance) If $(T, S), T', S) \in \Omega$ are such that, for all $i \in \{1, 2\}$, and for all quasi-order, $\geq_i$, generated by R_i, $T \geq_i T'$ and $T' \geq_i T$, then $f(T, S) = f(T', S)$.

The next axiom is motivated by the fact that the threat set is likely to be the set of payoffs the players earn from rationalizable solutions (Bernheim, 1984; Pearce, 1984) or from curb solutions (Basu and Weibull, 1991). Suppose the only two payoffs that occur from the rationalizable solutions of a normal-form game are $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x \gg y$. That is, in a normal-form game, if s is a strategy pair which is rationalizable and Π_i is player i's payoff function, then $(\Pi_1(s), \Pi_2(s))$ is either equal to x or y. And for each x and y, there exists a rationalizable strategy pair, s, such that $(\Pi_1(s), \Pi_2(s))$ is equal to it. It is arguable that if the two players find that their bargains fail and they have to play this game noncooperatively, then they will expect a payoff of x. That is, x is in some sense 'focal' within the set of payoffs possible under rationalizable solutions. In further support of this, check that if x and y are the only two payoffs that occur from rationalizable solutions and $x \gg y$, then there must exist a Nash equilibrium s^*, which gives a payoff of x and there does not exist a Nash equilibrium which Pareto dominates s^*. This is what motivates our next axiom. The next section shows how this axiom may be circumvented, should one be so inclined.

Axiom F (Focal Point) : If $(\{x, y\}, S) \in \Omega$ and $x \gg y$, then $f(\{x, y\}, S) = f(\{x\}, S)$.

Call a solution, f, a *max-max solution* if, for all $(T, S) \in \Omega$, $f(T, S)$ is the Kalai-Smorodinsky solution of the bargaining game $(\{\text{1-max } T,\ \text{2-max } T)\}, S)$. More formally let $(T, S) \in \Omega$. Define $\bar{S} := \{x \in S | x \geq (\text{1-max } T,\ \text{2-max } T)\}$ and L as the set containing all convex combinations of $(\text{1-max } T,\ \text{2-max } T)$ and $(\text{1-max } \bar{S},\ \text{2-max } \bar{S})$. If f is a max-max solution then $f(T, S)$ is $x \in L \cap \bar{S}$ such that, for all $y \in L \cap \bar{S}, x \geq y$. Figure 1 illustrates a max-max solution.

Figure 1 here

Therorem 1 : A solution satisfies axioms I, P, S, A, M, T and F if and only if it is the max-max solution.

Proof: Let f be a solution satisfying axiom I, P, S, A, M, T and F. Consider $(T, S) \in \Omega$. Let $\geq_i$ be a quasi-ordering generated by R_i, where (as before) we define R_i as follows. For all $x, y \in \mathbb{R}^2, xR_iy \leftrightarrow x_i \geq y_i$. From Lemma 1 we know that, for all $X, Y \subset \mathbb{R}^2$, $x \sim_i Y$ if 1-max X = 1-max Y and 1-min X = 1-min Y. It follows that, for all $X, Y \subset \mathbb{R}^2$, $X \sim_1 Y$ and $X \sim_2 Y$ if $H(X) = H(Y)$.

Define $x :=$ (1-max T, 2-max T) and $y :=$ (1-min T, 2-min T). Clearly $H(\{x, y\}) = H(T)$. Hence, the observation in the above paragraph and axiom T imply

$$f(\{x, y\}, S) = f(T, S).$$

In the light of axiom F this implies, $f(\{x, y\}, S) = f(\{x\}, S)$.

Hence $f(T, S) = f($1-max T, 2-max $T)\}, S)$.

Consider now a restriction of f to the domain of traditional bargaining games, $\overline{\Omega}$. Denote the restriction by $f|\bar{\Omega}$. Since $f|\bar{\Omega}$ satisfies axioms I, P, S, A and M, which are the Kalai-Smorodinsky axioms, it follows that $f|\bar{\Omega}$ is the Kalai-Smorodinsky solution. Since $f(\{x\}, S) = f|\bar{\Omega}(\{x\}, S)$, f is the max-max solution.

It is easy to check that the max-max solution, call it f^*, satisfies axioms I, P, S, A, M and F. To see that f^* satisfies axiom T, consider games $(D, S), (D', S) \in \Omega$. If $H(D) = H(D')$, axiom T would require that the solutions of (D, S) and (D', S) be the same. Since $f^*(D, S) = f^*(D', S)$, f^* satisfies this.

Suppose $H(D) \neq H(D')$. Without loss of generality, assume that (1-min D, 1-max D) $\neq$ (1-min D', 1-max D'). If 1-min $D >$ 1-min D', then from

Remark 1, following Lemma 2, we know that there exists a binary relation $\geq *$ generated by R_1 such that $D >* D'$. Thus axiom T is trivially satisfied. The case of 1-min $D <$ 1-min D' is symmetrically handled.

If 1-max $D >$ 1-max D' then from the proof of Lemma 2 we know that there exists a binary relation $\geq$ generated by R_1 such that $D > D'$. Thus T is again satisfied. The case of 1-max $D <$ 1-max D' is symmetrically handled.

■

The method of this proof makes it clear that if axiom M was replaced with the axiom of "independence of irrelevant alternatives", R, as defined below, then we could get the "max-Nash solution" : The solution f is *max-Nash* if, for all $(T,S) \in \Omega$, $f(T,S)$ is the Nash-bargaining solution of $(\{1\text{-max } T, 2\text{-max } T)\}, S)$.

Axiom R (Independence of Irrelevant Alternatives): Let $(\{d\},S)$, $(\{d\},S') \in \bar{\Omega}$. If $S' \subset S$ and $f(\{d\},S) \in S'$, then $f(\{d\},S') = f(\{d\},S)$.

Hence now we have the following theorem, which, being an analogue of Theorem 1, will not be proved here.

Theorem 2 : A solution satisfies axioms I, P, S, A, R, T and F if and only if it is the max-Nash solution.

5 Models without the Focal-Point Axiom

The axiom which to me seems to be the least acceptable in the above analysis is axiom F. Fortunately, the max-max solution can be reached by another route which does not use axiom F. But instead of doing this, I want to first axiomatize what I shall call the 'rectangular general solution'. Once this is worked out, it will be obvious what the alternative route to the max-max solution, which refrains from using axiom F, is.

The rectangular general solution has the advantage of applying to all general bargaining games. This means, in particular, that it applies to one very important game - one where the feasible set coincides with the disagreement set, that is, for a general bargaining game (T, S), where $T = S$. I shall describe such a game as a *bargaining game with complete uncertainty*. This is a particularly important game for the following reason. What is set S ? Under one standard interpretation, S is the set of all payoffs that can occur if some underlying (non-cooperative) game is played non-cooperatively[4]. One interpretation of the disagreement set, T, in a cooperative game is that it is the set of all payoffs that can occur if some cooperative bargain fails. Now, if a cooperative bargain fails, it is reasonable to suppose that the underlying game will have to be played non-cooperatively, which means that the set of possible payoffs in the event of disagreement, is the set of payoffs that can occur in the non-cooperative play of the game. Thus $T = S$. Hence this case seems important enough to deserve special attention.

To appreciate this further let us analyse the normal-form game, G, described below.

Game G

		Player 2	
		L	R
Player 1	U	1, 0	1, 2
	D	0, 1	2, 0

This game has no pure-strategy Nash equilibrium. The mixed-strategy Nash equilibrium results in the expected payoff pair $(1, \frac{2}{3})$. The feasible set of payoffs is depicted in Figure 2 by the $ABCD$.

Figure 2 here

Suppose the two players decide to play G cooperatively. Their feasible set is then given by $ABCD$. What payoff can they reasonably expect if they fail to reach a cooperative solution? A large body of literature would suggest that they should expect the Nash equilibrium payoffs, $(1, \frac{2}{3})$. One would then apply the Nash bargaining solution, Kalai-Smorodinsky solution or whatever else one wishes to the bargaining game $(\{(1, \frac{2}{3})\}, ABCD)$.

It is however not clear to me that $(1, \frac{2}{3})$ is a reasonable expectation in the event of disagreement, because this requires 2 to expect that player 1 will play U with probability $\frac{1}{3}$. But since in the Nash equilibrium, player 1 will be indifferent between U and D, there seems little ground to believe that 1 will mix U and D in proportions $\frac{1}{3}$ and $\frac{2}{3}$. This is one of the main problems that motivate the solution concept, curb, examined in Basu and Weibull (1991). In game G, curb coincides with rationalizability and predicts that any thing in $ABCD$ can happen. It seems right, therefore, to claim that the best way to analyse the *cooperative* solution of G is to assume that $ABCD$ is both the feasible set and the disagreement set.

Of course there may be other games where the disagreement set may reasonably be taken to consist of a unique point. The advantage of the solution concept I am about to describe and axiomatize is that it can solve both these kinds of bargaining games.

Return to the analysis of Section 3. As before, let U be the set of alternatives, $V := 2^u \backslash \{\phi\}$, R an ordering over U and $\geq$ a quasi-ordering over V. The asymmetric and symmetric parts of these relations are denoted by the usual notation. Consider the following new properties.

Property 4 : For all $A, B \in V$, if, for all $(x, y) \in A \times B$, xPy , then

$A > A \cup B$.

Property 5 : For all $A, B \in V$ and $x \in U$, if $A > B$ and xPy, for all $y \in A \cup B$, then $A \cup \{x\} > B \cup \{x\}$.

We shall now analyse the consequence of adding properties 4 and 5 to properties 1-3. Of course there are other properties, especially those which are symmetric to properties 4 and 5 which look as appealing as 1-5.

I shall comment on some of these later but in the meantime, 4 and 5 do look reasonable requirements, the consequences of which ought to be examined.

Let us return to the bargaining problem. Let Ω^* be the set of all general bargaining games. A *general solution* is a mapping $f : \Omega^* \to \mathbb{R}^2$ such that, for all $(T, S) \in \Omega^*$, $f(T, S) := (f_1(T, S), f_2(T, S)) \in S$. Clearly, if f is a general solution, the restriction of f to Ω, is a solution.

Next consider the following modification of axiom T. As before, let R_i be a binary relation on $\mathbb{R}^2$ such that, for all $x, y \in \mathbb{R}^2$, $xR_iy \leftrightarrow x_i \geq y_i$. We shall say that the ordering, $\geq_i$, on V (the collection of all non-empty subsets $\mathbb{R}^2$) *represents* R_i if $\geq_i$ satisfies properties 1-5.

Axiom T* : If $(T, S), (T', S) \in \Omega^*$ are such that, for all $i \in \{1, 2\}$ and for all order $\geq_i$, representing R_i, $T \sim_i T'$, then $f(T, S) = f(T', S)$, where f is a general solution.

A general solution, f, is a *rectangular general solution* if, for all $(T, S) \in \Omega^*, f(T, S)$ is the Kalai-Smorodinsky solution of the game $(\{(1\text{-min } T, \ 2\text{-min } T)\}), S)$. If for all $(T, S) \in \Omega^*$, $f(T, S)$ is the Nash bargaining solution of the game $(\{(1\text{-min } T, 2\text{-min } T)\}, S)$ we shall say that f is the *min-Nash general solution.*

And, finally, here is the main theorem of this section.

Theorem 3 : A general solution satisfies axioms I, P, S, A, M and T^* if and only if it is the rectangular general solution.

Proof : It is useful to first establish a preliminary result. Let R be the binary relation on $I\!R$ defined as follows : for all $x, y \in I\!R, xRy \leftrightarrow x \geq y$. Let $\bar{V}$ be the collection of all non-empty subsets of $I\!R$ and $\bar{Z}$ the collection of the compact elements of $\bar{V}$. The following will be proved:

(i) There exists an ordering $\geq$ on $\bar{V}$ which represents R.

(ii) If $\geq$ is an ordering on $\bar{V}$ which represents R, then, for all $A, B \in \bar{Z}, [A \sim B] \leftrightarrow [\min\ A = \min\ B]$.

To prove (i) define $\geq *$ on $\bar{V}$ as follows : for all $A, B, \in \bar{V}, [A \geq *B] \leftrightarrow [\forall\ x \in A, \exists\ y \in B$ such that $xRy]$. Clearly $\geq *$ is an ordering. From Remark 1 we know that it satisfies Properties 1-3. That it satisfies Properties 4 and 5 is obvious. Thus $\geq *$ represents R.

To prove (ii) suppose $\geq$ represents R. Let $A, B \in \bar{Z}$, and $\min\ A = \min\ B$. If $\max\ A = \max\ B$, it follows from Lemma 1 that $A \sim B$. Suppose $\max\ A > \max\ B$. Since from Lemma 1 we know that $\forall\ D \in \bar{Z}, D \sim \{\min\ D, \max\ D\}$, hence, without loss of geneality, assume that $A = [\min\ A, \max\ A]$ and $B = [\min\ B, \max\ B]$. Define $C = A \backslash B$. Hence, $A = C \cup B \geq B$, by property 1. There are now two possibilities. $A > B$ or $A \sim B$. Assume $A > B$. Choose $x \in R$ such that $xPy, \forall\ y \in A$. By property 5, $A \cup \{x\} > B \cup \{x\}$. Clearly $\max\ A \cup \{x\} = \max\ B \cup \{x\} = x$ and $\min\ A \cup \{x\} = \min\ B \cup \{x\}$. Hence, by Lemma 1, $A \cup \{x\} \sim B \cup \{x\}$. This contradiction proves that $A \sim B$.

To prove the reverse implication of (ii), suppose $\min\ A \neq \min\ B$. Without loss of generality, assume $\min\ A > \min\ B$. As before, without loss of generality also assume $A = [\min\ A, \max\ A]$ and $B = [\min\ B, \max\ B]$. Suppose $A \sim B$. Choose $x \in I\!R$ such that xPy, for all $y \in A \cup B$. By property 2, $A \cup \{x\} \sim B \cup \{x\}$. Let $\hat{A}$ and $\hat{B}$ be the smallest intervals containing, respectively, $A \cup \{x\}$ and $B \cup \{x\}$. By Lemma 1, $\hat{A} \sim \hat{B}$. Note that, for all $(x, y) \in \hat{A} \times (\hat{B} \backslash \hat{A}), xPy$. Thus, by Property 4, $\hat{A} > \hat{B}$. This contradiction establishes that $A \sim B$ is false. This completes the proof of (ii).

Let us now assume that f^* is a general solution satisfying axioms I, P, S, A, M and T^*. Consider any $(T, S) \in \Omega^*$. Let $\geq_i$ be an ordering on V which represents R_i. It will be shown that $T \sim_i \; i - \min \; T, i \in \{1, 2\}$. Without loss of generality focus on player 1. For all $A \in Z$, define $\underline{A} = \{x \in I\!R^2 | \exists \; y \in A$ such that $x_1 = y_1$ and $x_2 = 0\}$. Lemma 3 implies that, for all $A \in Z, A \sim_1 \underline{A}$. It immediately follows from (ii), above, that for all $T, T' \in Z, [T \sim_1 T'] \leftrightarrow$ [1-min $T =$ 1-min T']. Hence,

$$T \sim_i \{(1 - \min \; T, \;\; \text{2-min} \; T)\}, \text{ for all } \; i \in \{1, 2\}.$$

Axiom T^* implies that $f^*(T, S) = f^*(\{(\text{1-min } T, \;\; \text{2-min } T)\}, S)$.

If $f^*|\bar{\Omega}$ is the restriction of f^* on $\bar{\Omega}$, it follows from the fact that $f^*|\bar{\Omega}$ satisfies axioms I, P, S, A and M, that $f^*|\bar{\Omega}$ is the Kalai-Smorodinsky solution of the game $(\{(\text{1-min } T, \;\; \text{2-min } \; T)\}, S)$. Thus f^* is the rectangular general solution.

To prove the converse part of theorem 3, assume $\hat{f}$ is the rectangular general solution. Hence $\hat{f}|\bar{\Omega}$ is the Kalai-Smorodinsky solution. Thus $\hat{f}$ satisfies axioms I, P, S, A and M. To establish axiom T^*, note that we have proved above (see (ii)) that, for all $T, T' \in Z$,

$$[T \sim_i T'] \rightarrow [\text{i-min } T = \text{i-min } T'], \;\; \text{for } \; i \in \{1, 2\},$$

whenever $\geq_i$ represents R_i. From the definition of $\hat{f}$, we know that :

$$[\text{i-min } T = \text{i-min } T', i \in \{1, 2\}] \rightarrow [\hat{f}(T, S) = \hat{f}(T', S)].$$

Thus $\hat{f}$ satisfies axiom T^*. ■

Return to the example depicted in Figure 2. The general bargaining game depicted there has both the feasible set and the disagreement set given by $ABCD$. If we want to solve the bargaining problem while satisfying axioms I, P, S, A, M and T^*, Theorem 3 assests that the two players must earn the payoff $(1\frac{1}{3}, 1\frac{1}{3})$, as illustrated.

Following the same logic as in the discussion following theorem 1, we have the following obvious corollary of Theorem 3.

Theorem 4 : A general solution satisfies axioms I, P, S, A, R and T^* if and only if it is the min-Nash generalsolution.

It was stated at the beginning of this section that the max-max solution can be axiomatized without the use of the focal-point axiom. As must be obvious by now, this can be achieved by modifying properties 4 and 5. Thus, for instance in property 4 instead of requiring $A > A \cup B$ we need to demand $A \cup B > B$. Similarly in property 5 instead of using an x that dominates all the elements of A and B we need to work with an x that is dominated by all the elements of A and B.

If instead of modifying properties 4 and 5 we impose the requirements in the above paragraph as additional demand on $\geq_1$ and $\geq_2$, (that is, we require that 4, 5 and their variants just suggested be satisfied), then we end up with an obvious impossibility theorem.

I do not want to prejudge between properties 4 and 5 and their variants, since the objective of this paper is to bring out the implications of different axioms and what they imply in terms of solutions of bargaining games, the final choice being left to the reader. While the formal implications of properties 4 and 5 and their variants were stated above, their intuitions are best understood in terms of player pessimism and optimism. Consider an extremely pessimistic person, who, when he gets a set, X, of alternatives, expects to end up with the worst element in X. Now suppose A and B are such that, for all $x \in A$ and for all $y \in B, xPy$, then clearly the extreme pessimist will prefer A to $A \cup B$ but be indifferent between B and $A \cup B$ since the worst elements of the last two sets are the same. Hence, property 4 may be viewed as apt when the players are pessimistic; and, by an obverse argument, the variant of property 4, discussed above, is apt when the players are optimistic.

6 Concluding Remarks

The aim of this paper was to expand the scope of two-player bargaining problems to cases where the standard threat point is replaced with a threat set. This brings into discourse a variety of new axioms and problems connected with ranking *sets* of alternatives. As stressed at the end of the last section, my objective here is to draw out the implications of different axioms without, at this early stage, trying to make any strong case for one over the other.

Many of the results here come in pairs - the Kalai-Smorodinsky-*type* solution and the Nash-*type* solution depending on whether the new axioms developed here are combined with, respectively, the monotonicity axiom or the axiom of independence of irrelevant alternatives. While my own preference is for the monotonicity axiom this is not germane to the concerns of this paper.

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Endnotes

1. The case where the probabilities of different possible threat points are known has been recently modeled (see, for instance, Chun and Thomson, 1989).

2. Indeed, if the failure of bargaining reveals some player's irrationality, then even more complicated questions arise which are akin to the 'problem of unreached nodes' in extensive form game theory. I have discussed this in Basu (1990).

3. Given a normal-form game G, a curb set, T, of strategies is a compact subset of the Cartesian product of the two players' sets of strategies such that the best response to T is a subset of T. T is tight curb if the best response to it is T. A curb set, may be viewed as a set-valued counterpart of a strict equilibrium. Given that for some games, the payoff in a curb set could Pareto-dominate the payoff earned at some Nash equilibrium, it may be reasonable in some games to expect players to play within a curb set.

4. Though in the example that follows I allow for joint randomization of strategies, this is by no means necessary for my argument here. All that we need is that the kind of randomizations that is allowed in constructing feasible sets be allowed in constructing threat sets.

Figure 1

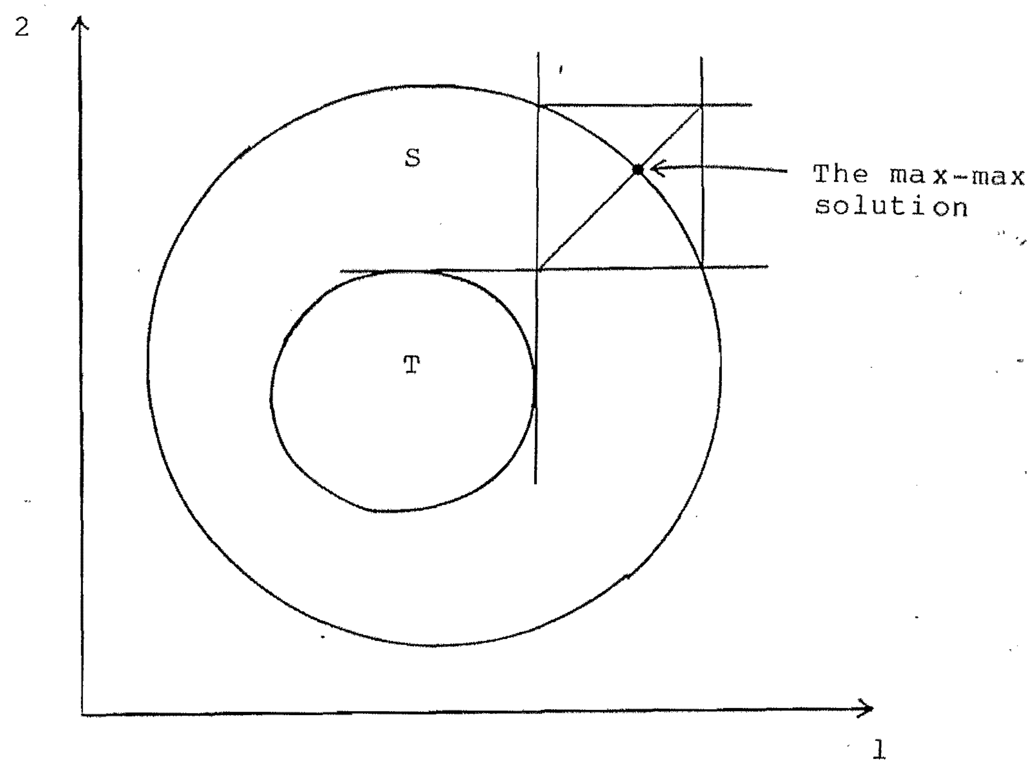
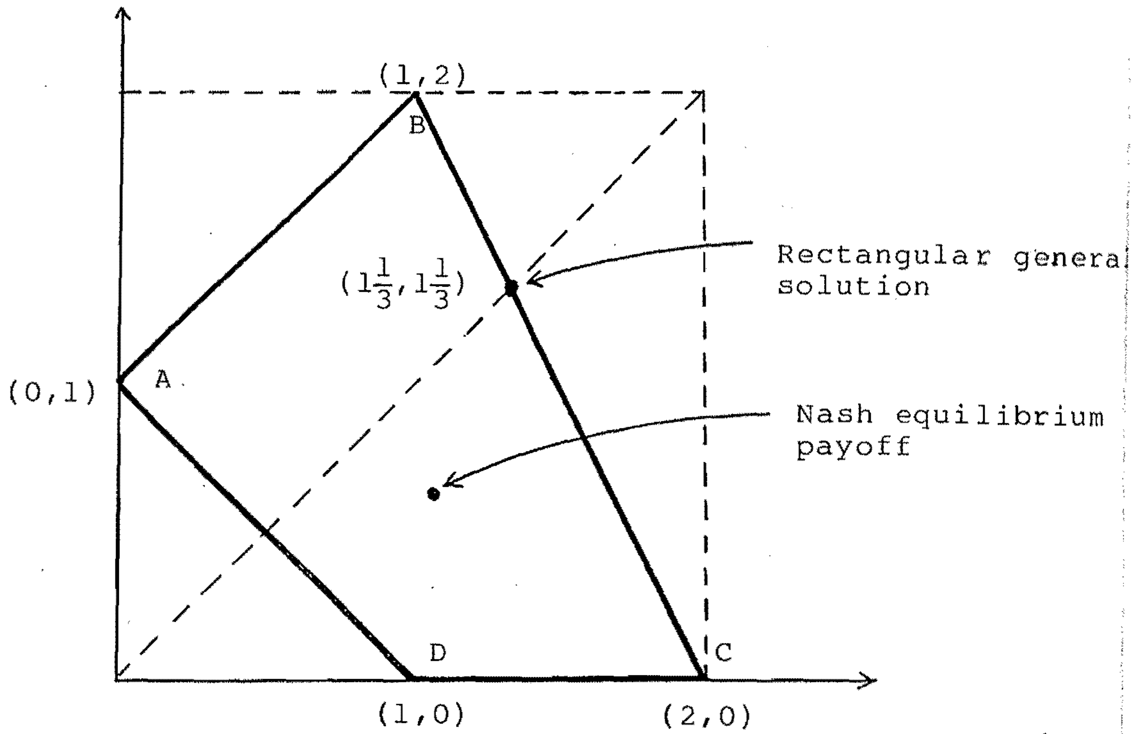


Figure 2


(1,2)
B
Rectangular general
solution
(1$\frac{1}{3}$,1$\frac{1}{3}$)
(0,1)
A
Nash equilibrium
payoff
D
C
(1,0)
(2,0)

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