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A NOTE ON RANDOMIZED SOCIAL CHOICE AND RANDOM DICTATORSHIPS

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ABSTRACT

We show that rationalizability can be replaced by the weaker regularity condition in McLennan's [6] random dictatorship result for decision super-schemes. Our result also shows that, when there are at least three alternatives in the universal set, the independence of irrelevant alternatives condition of Pattanaik and Peleg [7] together with their requirement that there be at least two more alternatives in the universal set than there are individuals in the society can be replaced by strategy proofness to obtain an alternative characterization of random dictatorships.

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1 Introduction

In the paper "Randomized Preference Aggregation: Additivity of Power and Strategy Proofness", which was published in the *Journal of Economic Theory* (1980), McLennan [6] looked at *decision super-schemes* that satisfy three axioms, namely, *rationalizability*, *strategy proofness* and *Pareto optimific ex-post*. Decision super-schemes are randomized collective decision rules that map each combination of a social preference profile and a feasible set of alternatives to a social choice lottery over the feasible set. A decision super-scheme is rationalizable if, for each preference profile, there is a probability distribution over social preferences that induces the social choice lottery over each feasible set. A decision super-scheme is strategy proof if no one can misrepresent her preference and increase her expected utility from the social choice lottery over any feasible set according to any von Neumann–Morgenstern utility function that represents her true preference. A decision super-scheme is Pareto optimific ex-post if a feasible alternative has zero probability of being chosen whenever there is another feasible alternative that everyone prefers to it. When individual preferences are restricted to strict orderings, McLennan showed that any decision super-scheme that is rationalizable, strategy proof and Pareto optimific ex-post must be a random dictatorship. A decision super-scheme is said to be randomly dictatorial if there is a vector of individual weights which has the properties of a probability distribution over the set of all individuals and the social probability of choosing an alternative from a feasible set is equal to the sum of the weights of those individuals who have that alternative as their best alternative in the feasible set.

The purpose of this note is to point out that, although McLennan [6] uses rationalizability to prove his result, this condition can be replaced by another collective rationality condition known as *regularity* without affecting the result. It has been known for some time that regularity by itself is a weaker condition than rationalizability.[1] Regularity is an appropriate analogue of Chernoff's [4] property α, which is a minimum consistency condition for rationalizability of deterministic choice functions, and requires that, given the same preference profile, the probability of choosing an alternative from a feasible set must not increase from its original value when the feasible set expands.

This note also provides an answer to a question related to the work of Pattanaik and Peleg [7]. When individuals only have strict preference orderings, Pattanaik and Peleg essentially characterized random dictatorships as those decision super-schemes that are regular, Pareto optimific ex-post, and satisfy *independence of irrelevant alternatives.*[2] Independence of irrelevant alternatives requires that the social choice lottery over a feasible set must be the same for any two preference profiles in which the individual preferences restricted to the feasible set are the same. In their characterization of random dictatorships, Pattanaik and Peleg required two other conditions to hold in addition to their three axioms. They gave examples to show that their result may no longer be true without these two additional conditions. The first condition requires that the universal set of social alternatives must contain at least four alternatives. The second condition, which is needed for their result to hold when the universal set itself is the feasible set, says that the number of alternatives in the universal set must exceed the number of individuals in the society by at least two. This naturally opens up the question of whether there is a reasonable system of axioms that completely characterizes random dictatorships when the additional conditions of Pattanaik and Peleg are relaxed. The result presented in this note provides a positive response to this question. We show that random dictatorships can be completely characterized without the additional conditions of Pattanaik and Peleg by replacing the independence of irrelevant alternatives axiom with strategy proofness provided there are at least three alternatives in the universal set.

[1]For example, see Lemma 3.13 of Pattanaik and Peleg [7].

[2]It must be pointed out that Pattanaik and Peleg [7] uses the terms *probabilistic voting procedure* and *Paretian ex-post* for decision super-scheme and Pareto optimific ex-post, respectively.

2 Preliminaries

There are n individuals in the society and m elements in the universal set of social alternatives. We denote the society by N $(= \{1, ..., n\})$ and the universal set by A, with $\infty > |N| = n \geq 2$ and $\infty > |A| = m \geq 2$.[3] Also, we denote the set of all nonempty subsets of A by $\mathcal{A}$ and the set of all *linear orderings* on A by $\mathcal{L}$. A linear ordering on A is a reflexive, complete, transitive and antisymmetric binary relation on A.[4]

The n-fold Cartesian product of $\mathcal{L}$ is denoted by $\mathcal{L}^N$. We use the term *preference profile* for the members of $\mathcal{L}^N$ and denote them by $\mathbf{P}$, $\hat{\mathbf{P}}$, $\tilde{\mathbf{P}}, \ldots$. For each $\mathbf{P} \in \mathcal{L}^N$, the ith coordinate of $\mathbf{P}$, which we denote by P_i, is the preference relation of individual i in the preference profile $\mathbf{P}$.

Definition 1: A *decision super-scheme* (DSS) is a function $F : A \times \mathcal{A} \times \mathcal{L}^N \rightarrow \Re_+$ such that:

$$\sum_{x \in B} F(x, B, \mathbf{P}) = \sum_{x \in A} F(x, B, \mathbf{P}) = 1 \quad \text{for every } (B, \mathbf{P}) \in \mathcal{A} \times \mathcal{L}^N.$$

For any DSS F, given a feasible set B, a preference profile $\mathbf{P}$ and an alternative x, $F(x, B, \mathbf{P})$ represents the probability of x being chosen by the society when the feasible set is B and society's preference profile is $\mathbf{P}$. Thus, a DSS assigns zero probability to any alternative which does not belong to the feasible set.

Given a DSS F and any $(B, \mathbf{P}) \in \mathcal{A} \times \mathcal{L}^N$, let

$$\begin{aligned} PAR(B, \mathbf{P}) &= \{x \in B : \text{ there does not exist } y \in B \text{ such that } yP_ix \text{ for all } i \in N\} \quad \text{and} \\ POS(F, B, \mathbf{P}) &= \{x \in B : \ F(x, B, \mathbf{P}) > 0\}. \end{aligned}$$

So $PAR(B, \mathbf{P})$ is the set of all Pareto optimal alternatives in B according to $\mathbf{P}$, and $POS(F, B, \mathbf{P})$ is the set of all alternatives that are assigned positive probabilities by F when B is the feasible set and $\mathbf{P}$ is the preference profile.

[3] For any set D, we denote the number of elements in D by $|D|$.

[4] A binary relation R is antisymmetric if xRy for any two distinct alternatives x and y implies that yRx cannot hold.

Definition 2: A DSS F is *Pareto optimific ex-post* (POEP) with respect to a $B \in \mathcal{A}$ if $POS(F,B,\mathbf{P}) \subseteq PAR(B,\mathbf{P})$ for every $\mathbf{P} \in \mathcal{L}^N$. A DSS F is POEP if it is POEP with respect to every $B \in \mathcal{A}$.

Suppose $B \in \mathcal{A}$. For each $i \in N$ and for each $P_i \in \mathcal{L}$, we denote the restriction of P_i to B by $P_i|B$. Similarly, for each $\mathbf{P} \in \mathcal{L}^N$, we denote the restriction of $\mathbf{P}$ to B by $\mathbf{P}|B = (P_1|B, ..., P_n|B)$. Then the ***independence of irrelevant alternatives*** (IIA) axiom is defined as follows.

Definition 3: A DSS F satisfies IIA if, for all $B \in \mathcal{A}$ and for all $\mathbf{P}, \hat{\mathbf{P}} \in \mathcal{L}^N$:

$$[\mathbf{P}|B = \hat{\mathbf{P}}|B] \implies [F(x,B,\mathbf{P}) = F(x,B,\hat{\mathbf{P}}) \text{ for all } x \in B]\,.$$

Given any $i \in N$ and any $P_i \in \mathcal{L}$, we denote the set of all von Neumann–Morgenstern utility functions that represent the preference relation P_i by $\mathcal{U}(P_i)$. Also, given any $\mathbf{P} \in \mathcal{L}^N$ and any $i \in N$, we use the notation $\mathbf{P}_{-i} = (P_1, ..., P_{i-1}, P_{i+1}, ..., P_n)$ and $(\mathbf{P}_{-i}, \hat{P}_i) = (P_1, ..., P_{i-1}, \hat{P}_i, P_{i+1}, ..., P_n)$ for any $\hat{P}_i \in \mathcal{L}$. The ***strategy proofness*** axiom is then given as follows.

Definition 4: A DSS F is ***strategy proof*** (SP) at a $B \in \mathcal{A}$ if, given any $\mathbf{P} \in \mathcal{L}^N$, for each $i \in N$ and every $U \in \mathcal{U}(P_i)$:

$$\sum_{x \in B}[F(x,B,\mathbf{P})U(x)] \;\geq\; \sum_{x \in B}[F(x,B,(\mathbf{P}_{-i},\hat{P}_i))U(x)] \quad \text{for all } \hat{P}_i \in \mathcal{L}\,.$$

A DSS F is SP if it is SP at every $B \in \mathcal{A}$.

Given any $B \in \mathcal{A}$ and any $x \in B$, let

$$q(x,B) \;=\; \{P \in \mathcal{L} : \; xPy \text{ for every } y \in B \text{ distinct from } x\}.$$

The two collective rationality conditions, namely, ***rationalizability*** and ***regularity***, are then defined as follows.

Definition 5: A DSS F is ***rationalizable*** if, corresponding to each $\mathbf{P} \in \mathcal{L}^N$, there exists a probability distribution $r_{\mathbf{P}}$ over $\mathcal{L}$ such that, for every $B \in \mathcal{A}$:

$$F(x,B,\mathbf{P}) \;=\; \sum_{P \in q(x,B)} r_{\mathbf{P}}(P) \quad \text{for all } x \in B.$$

Definition 6: A DSS F is *regular* if, for all $B, \hat{B} \in \mathcal{A}$ and for all $\mathbf{P} \in \mathcal{L}^N$:

$$[x \in B \subseteq \hat{B}] \implies [F(x, B, \mathbf{P}) \geq F(x, \hat{B}, \mathbf{P})] .$$

As noted earlier, it is well known that a rationalizable DSS is also regular, but a regular DSS is not necessarily rationalizable.

We need one more notation to introduce the notion of a randomly dictatorial DSS. Given any $B \in \mathcal{A}$, any $x \in B$ and any $\mathbf{P} \in \mathcal{L}^N$, we denote by $L(x, \mathbf{P}|B)$ the set of all individuals who have x as their most preferred alternative in B according to their preferences in $\mathbf{P}$; i.e.

$$L(x, \mathbf{P}|B) = \{i \in N : \ xP_iy \text{ for every } y \in B \text{ distinct from } x\} .$$

Definition 7: A DSS F is a *random dictatorship* at a $B \in \mathcal{A}$ if each $i \in N$ has an individual weight $\alpha_i^B \in [0, 1]$ such that:

(i) $\sum_{i \in N} \alpha_i^B = 1$; and

(ii) $F(x, B, \mathbf{P}) = \sum_{i \in L(x, \mathbf{P}|B)} \alpha_i^B$ for all $x \in B$ and for all $\mathbf{P} \in \mathcal{L}^N$,

where we let $\sum_{i \in L(x, \mathbf{P}|B)} \alpha_i^B = 0$ if $L(x, \mathbf{P}|B) = \emptyset$. A DSS F is a *random dictatorship* if: (i) it is a random dictatorship at every $B \in \mathcal{A}$; and (ii) for every $B, \hat{B} \in \mathcal{A}$, $\alpha_i^B = \alpha_i^{\hat{B}}$ for all $i \in N$.

We now present formally the findings of McLennan [6], and Pattanaik and Peleg [7]. The interested reader is referred to the original articles for the proofs.

Proposition 1 (McLennan [6, Theorem 6]): *Suppose $m \geq 3$, and F is a* DSS *that is rationalizable,* POEP *and* SP. *Then F is a random dictatorship.*

Proposition 2 (Pattanaik and Peleg [7, Theorems 4.11 and 4.14]): *Suppose $m \geq n + 2$, and F is a* DSS *that is regular,* POEP *and satisfies* IIA. *Then F is a random dictatorship.*

3 Characterization

Our main objective in this section is to show that Proposition 1 holds even if the rationalizability requirement is replaced by the weaker regularity condition. To achieve this we need some straightforward observations and preliminary results.[5]

Observation 1: Suppose $B \in \mathcal{A}$, and $P, \hat{P} \in \mathcal{L}$. If $P|B = \hat{P}|B$ and $U \in \mathcal{U}(P)$, then there exists $\hat{U} \in \mathcal{U}(\hat{P})$ such that $U(x) = \hat{U}(x)$ for all $x \in B$.

Observation 2: Suppose $P \in \mathcal{L}$, $U \in \mathcal{U}(P)$, and $y \in A$. Then there exists $\hat{U} \in \mathcal{U}(P)$ such that $U(x) = \hat{U}(x)$ for every $x \in A$ distinct from y, and $U(y) > \hat{U}(y)$.

Using the two observations given above, we prove the following lemma.

Lemma 1 : *If a* DSS F *is* SP, *then* F *satisfies* IIA.

Proof: Let F be a DSS that is SP. But suppose F does not satisfy IIA. Then there exists $B \in \mathcal{A}$ and $\mathbf{P}, \hat{\mathbf{P}} \in \mathcal{L}^N$, with $\mathbf{P}|B = \hat{\mathbf{P}}|B$, such that $\{F(x, B, \mathbf{P})\}_{x \in B} \neq \{F(x, B, \hat{\mathbf{P}})\}_{x \in B}$. Consider the following preference profiles which are obtained from $\mathbf{P}$ by successively replacing the individual preferences in it by the corresponding individual preferences in $\hat{\mathbf{P}}$: $\mathbf{P}^1 = (\hat{P}_1, P_2, ..., P_n)$, $\mathbf{P}^2 = (\hat{P}_1, \hat{P}_2, P_3, ..., P_n)$, ..., $\mathbf{P}^{n-1} = (\hat{P}_1, ..., \hat{P}_{n-1}, P_n)$, and $\mathbf{P}^n = \hat{\mathbf{P}}$. As $\{F(x, B, \mathbf{P})\}_{x \in B} \neq \{F(x, B, \hat{\mathbf{P}})\}_{x \in B}$, there exists some $2 \leq j \leq n$ such that $\{F(x, B, \mathbf{P}^{j-1})\}_{x \in B} \neq \{F(x, B, \mathbf{P}^j)\}_{x \in B}$. Let $U \in \mathcal{U}(P_j)$. Then we look at two mutually exclusive cases that exhaust all possibilities:

(I) $\sum_{x \in B}[F(x, B, \mathbf{P}^{j-1})U(x)] \leq \sum_{x \in B}[F(x, B, \mathbf{P}^j)U(x)]$:

As $\{F(x, B, \mathbf{P}^{j-1})\}_{x \in B} \neq \{F(x, B, \mathbf{P}^j)\}_{x \in B}$, let $z \in B$ be such that $F(z, B, \mathbf{P}^{j-1}) < F(z, B, \mathbf{P}^j)$. Also, using Observation 2, let $\tilde{U} \in \mathcal{U}(P_j)$ be such that $\tilde{U}(x) = U(x)$ for all $x \in A$ distinct from z and $\tilde{U}(z) > U(z)$. Then it can be checked that

$$\sum_{x \in B}[F(x, B, \mathbf{P}^{j-1})\tilde{U}(x)] < \sum_{x \in B}[F(x, B, \mathbf{P}^j)\tilde{U}(x)].$$

[5] The proofs of Observations 1 and 2 can be made available on request.

Therefore, when everyone except j report their respective preferences in $\mathbf{P}^j_{-j}$ and j's true preference is P_j, j's expected utility from the social choice lottery over B according to $\hat{U} \in \mathcal{U}(P_j)$ is higher if she reports $\hat{P}_j$ instead of P_j. This contradicts our assumption that F is SP.

(II) $\sum_{x \in B}[F(x, B, \mathbf{P}^{j-1})U(x)] > \sum_{x \in B}[F(x, B, \mathbf{P}^j)U(x)]$:

Using $P_j|B = \hat{P}_j|B$ and Observation 1, let $\hat{U} \in \mathcal{U}(\hat{P}_j)$ be such that $U(x) = \hat{U}(x)$ for all $x \in B$. Then

$$\sum_{x \in B}[F(x, B, \mathbf{P}^{j-1})\hat{U}(x)] > \sum_{x \in B}[F(x, B, \mathbf{P}^j)\hat{U}(x)].$$

This means that, when everyone except j report their respective preferences in $\mathbf{P}^j_{-j}$ and j's true preference is $\hat{P}_j$, j's expected utility from the social choice lottery over B according to $\hat{U} \in \mathcal{U}(\hat{P}_j)$ is higher if she reports P_j instead of $\hat{P}_j$. Again a contradiction to our assumption that F is SP.

This completes the proof of the lemma. ||

The next lemma is a corollary to Gibbard's [5] result credited to H. Sonnenschein. Although it is stated somewhat differently using different terminology, Lemma 2 is essentially the same as Corollary 1 in Gibbard [5].

Lemma 2 : *Suppose F is a* DSS *that is* SP *and* POEP, *and $B \in \mathcal{A}$ is such that $|B| \geq 3$. Then F is a random dictatorship at B.*

Our last preliminary result is a lemma due to Pattanaik and Peleg [7]. This lemma uses the regularity axiom to characterize the condition under which the probability assigned by a DSS to each alternative in a feasible set remains unchanged when the feasible set is expanded.

Lemma 3 (Pattanaik and Peleg [7, Lemma 4.1]): *Suppose F is a regular* DSS, *and $B, \hat{B} \in \mathcal{A}$, with $B \subseteq \hat{B}$. Then, for any $\mathbf{P} \in \mathcal{L}^N$, $POS(F, \hat{B}, \mathbf{P}) \subseteq B$ if and only if $F(x, B, \mathbf{P}) = F(x, \hat{B}, \mathbf{P})$ for all $x \in B$.*

We now present our main characterization as Proposition 3.

Proposition 3 : *Suppose $m \geq 3$, and F is a* DSS *that is regular,* POEP *and* SP. *Then F is a random dictatorship.*

Proof: Let $m \geq 3$, and let F be a DSS that is regular, SP and POEP. By Lemma 2, F is a random dictatorship at A. Then it is sufficient to show that, given any $B \in \mathcal{A}$ and any $\mathbf{P} \in \mathcal{L}^N$:

$$F(x, B, \mathbf{P}) = \sum_{i \in L(x, \mathbf{P}|B)} \alpha_i^A \quad \text{for every } x \in B .$$

Consider any $B \in \mathcal{A}$ and any $\mathbf{P} \in \mathcal{L}^N$. Let $z \in B$. Also, let $\hat{\mathbf{P}} \in \mathcal{L}^N$ be such that $\hat{\mathbf{P}}|B = \mathbf{P}|B$, and, for each $i \in N$, $z\hat{P}_i y$ for all $y \notin B$. Then, as F is POEP, we get $POS(F, A, \hat{\mathbf{P}}) \subseteq B$. So Lemma 3 implies that

$$F(x, A, \hat{\mathbf{P}}) = F(x, B, \hat{\mathbf{P}}) \quad \text{for all } x \in B.$$

Because of Lemma 1 and the fact that F is SP, we know that F satisfies IIA. So, as $\mathbf{P}|B = \hat{\mathbf{P}}|B$, we also get

$$F(x, B, \hat{\mathbf{P}}) = F(x, B, \mathbf{P}) \quad \text{for all } x \in B.$$

Therefore, we have

$$F(x, A, \hat{\mathbf{P}}) = F(x, B, \mathbf{P}) \quad \text{for all } x \in B.$$

It is clear from our specification of $\hat{\mathbf{P}}$ that $L(x, \hat{\mathbf{P}}|A) = L(x, \hat{\mathbf{P}}|B)$ for every $x \in B$. As $\mathbf{P}|B = \hat{\mathbf{P}}|B$, we also know that $L(x, \hat{\mathbf{P}}|B) = L(x, \mathbf{P}|B)$ for every $x \in B$. So we have $L(x, \hat{\mathbf{P}}|A) = L(x, \mathbf{P}|B)$ for all $x \in B$. Thus, as F is a random dictatorship at A, it follows that

$$F(x, A, \hat{\mathbf{P}}) = \sum_{i \in L(x, \mathbf{P}|B)} \alpha_i^A \quad \text{for all } x \in B.$$

But we already know that $F(x, A, \hat{\mathbf{P}}) = F(x, B, \mathbf{P})$ for all $x \in B$. Hence, as desired, we have

$$F(x, B, \mathbf{P}) = \sum_{i \in L(x, \mathbf{P}|B)} \alpha_i^A \quad \text{for every } x \in B . \quad \|$$

As regularity by itself is a weaker condition than rationalizability, it is quite natural and appropriate to know whether the system of axioms stipulated in Proposition 1 is indeed more demanding than that stipulated in Proposition 3. When there are at least three alternatives in the universal set, it turns out that, once a DSS is required to be POEP and SP, requiring it to be regular is as strong as requiring it to be rationalizable. Thus, although it may not seem so, the systems of axioms in Propositions 1 and 3 are equivalent. This is a consequence of Proposition 4 given below.

Proposition 4 : *If a* DSS F *is a random dictatorship, then* F *is rationalizable,* POEP *and* SP.

Proof: Let F be a randomly dictatorial DSS with the vector of individual weights $(\alpha_1, ..., \alpha_n)$.

(Rationalizability): Consider any $\mathbf{P} \in \mathcal{L}^N$, and let $r_\mathbf{P}$ be the probability distribution over $\mathcal{L}$ such that $r_\mathbf{P}(P_i) = \alpha_i$ for each $i \in N$ and $r_\mathbf{P}(P) = 0$ for each $P \in \mathcal{L}$ distinct from all the individual preferences in the profile $\mathbf{P}$. Also, let $B \in \mathcal{A}$. Then it is sufficient to show that

$$F(x, B, \mathbf{P}) \;=\; \sum_{P \in q(x,B)} r_\mathbf{P}(P) \quad \text{for all } x \in B.$$

For each $x \in B$, one can verify that $P_i \in q(x, B)$ if $i \in L(x, \mathbf{P}|B)$. It then follows from the definition of $r_\mathbf{P}$ that

$$\sum_{P \in q(x,B)} r_\mathbf{P}(P) \;=\; \sum_{i \in L(x,\mathbf{P}|B)} r_\mathbf{P}(P_i) \;=\; \sum_{i \in L(x,\mathbf{P}|B)} \alpha_i \quad \text{for all } x \in B.$$

But we know that $F(x, B, \mathbf{P}) = \sum_{i \in L(x,\mathbf{P}|B)} \alpha_i$ for each $x \in B$ as F is a random dictatorship with the individual weights $(\alpha_1, ..., \alpha_n)$. Hence, the proof of rationalizability is complete.

(POEP): Consider any $\mathbf{P} \in \mathcal{L}^N$, and any $B \in \mathcal{A}$. If $x \in B$ is such that $F(x, B, \mathbf{P}) > 0$, then $L(x, \mathbf{P}|B)$ is nonempty as F is randomly dictatorial. Also, if $x \in B$ is such that $L(x, \mathbf{P}|B)$ is nonempty, then $x \in PAR(B, \mathbf{P})$. So, for any $x \in B$, $x \in POS(F, B, \mathbf{P})$ implies that $x \in PAR(B, \mathbf{P})$. Therefore, F is POEP.

(SP): Let $\mathbf{P} \in \mathcal{L}^N$, and let $B \in \mathcal{A}$. Consider any $i \in N$, any $\hat{P}_i \in \mathcal{L}$, and any $U \in \mathcal{U}(P_i)$. Then all that we need to show is that

$$\sum_{x\in B}[F(x,B,\mathbf{P})U(x)] \geq \sum_{x\in B}[F(x,B,(\mathbf{P}_{-i},\hat{P}_i))U(x)].$$

Let $y \in B$ be the best alternative in B according to P_i, and let $z \in B$ be the best alternative in B according to $\hat{P}_i$. Suppose $y = z$. Then $L(x,\mathbf{P}|B) = L(x,(\mathbf{P}_{-i},\hat{P}_i)|B)$ for every $x \in B$, and hence, $F(x,B,\mathbf{P}) = F(x,B,(\mathbf{P}_{-i},\hat{P}_i))$ for every $x \in B$ as F is a random dictatorship. So we have

$$\sum_{x\in B}[F(x,B,\mathbf{P})U(x)] = \sum_{x\in B}[F(x,B,(\mathbf{P}_{-i},\hat{P}_i))U(x)].$$

Now, suppose $y \neq z$. Then, as F is a random dictatorship with individual weights $(\alpha_1,...,\alpha_n)$, we get $F(y,B,(\mathbf{P}_{-i},\hat{P}_i)) = F(y,B,\mathbf{P}) - \alpha_i$, $F(z,B,(\mathbf{P}_{-i},\hat{P}_i)) = F(z,B,\mathbf{P}) + \alpha_i$ and $F(x,B,(\mathbf{P}_{-i},\hat{P}_i)) = F(x,B,\mathbf{P})$ for all $x \in B$ distinct from y and z. So

$$\sum_{a\in B}[F(a,B,\mathbf{P}) - F(a,B,(\mathbf{P}_{-i},\hat{P}_i))]U(a) = \alpha_i[U(y) - U(z)].$$

But, as y is the best alternative in B according to P_i and $U \in \mathcal{U}(P_i)$, it must be the case that $U(y) - U(z) > 0$. Therefore, we get $\alpha_i[U(y) - U(z)] \geq 0$. Hence, as required, we have

$$\sum_{a\in B}[F(a,B,\mathbf{P})U(a)] \geq \sum_{a\in B}[F(a,B,(\mathbf{P}_{-i},\hat{P}_i))U(a)].$$

This completes the proof of Proposition 4. ||

Thus, if a DSS is regular, POEP and SP, then it is also rationalizable as it must be a random dictatorship. So we cannot find any DSS that is regular, POEP and SP, but not rationalizable. Also, when $m \geq n+2$, as a DSS that is regular, POEP and satisfies IIA is a random dictatorship, it must be rationalizable. Therefore, in Pattanaik and Peleg [7], as soon as the additional condition $m \geq n+2$ is imposed to obtain a complete characterization of random dictatorships, requiring regularity is as strong as requiring rationalizability.

Proposition 4 also has an implication for the relationship between IIA and strategy proofness. As Pattanaik and Peleg [7] showed that Proposition 2 is not true if $m < n+2$, Lemma 1 and Proposition 3 imply that, even when a DSS is regular and POEP, requiring it to satisfy IIA is weaker than requiring it to be SP if $m < n+2$. However, when $m \geq n+2$, because of Proposition

4, the systems of axioms in Propositions 2 and 3 are equivalent. Therefore, when the number of alternatives in the universal set exceeds the number of individuals in the society by at least two, once a DSS is required to be regular and POEP, requiring it to satisfy IIA is no less demanding than requiring it to be SP.

4 Conclusion

In this note we have shown that the rationalizability axiom used by McLennan [6] to prove his random dictatorship result can be replaced by the weaker regularity axiom without invalidating his result. However, although it may not seem so, replacing rationalizability by regularity does not lead to a system of axioms that accomodates more DSSs. This is because, as soon as a DSS is required to be POEP and SP, imposing regularity on the DSS is as stringent as imposing rationalizability. Our result also showed that a possible way to keep the result of Pattanaik and Peleg [7] intact without the condition $m \geq n+2$ is to replace IIA by strategy proofness. However, when $m \geq n+2$ does hold, if we require a DSS to be POEP and satisfy IIA, then it does not matter whether we impose regularity or rationalizability.

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