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DUALITY MAPPINGS FOR THE THEORY OF RISK AVERSION WITH VECTOR OUTCOMES

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Duality mappings for the theory of risk aversion with vector outcomes

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Abstract

We consider a decision-making environment with an outcome space that is a convex and compact subset of a vector space belonging to a general class of such spaces. Given this outcome space, we define general classes of (a) risk averse von Neumann-Morgenstern utility functions defined over the outcome space, (b) multi-valued mappings that yield the certainty equivalent outcomes corresponding to a lottery, (c) multi-valued mappings that yield the risk premia corresponding to a lottery, and (d) multi-valued mappings that yield the acceptance set of lotteries corresponding to an outcome. Our duality results establish that the usual mappings that generate (b), (c) and (d) from (a) are bijective. We apply these results to the problem of computing the value of financial assets to a risk averse decision-maker and show that this value will always be less than the arbitrage-free valuation.

JEL classification: C02, D01, D81

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1 Introduction

The theory of risk aversion (Arrow [1], Pratt [19], Yaari [22]) is one of the most useful applications of the general theory of choice from the set of lotteries over an outcome space. This theory featuring real-valued outcomes characterizes risk aversion and comparative risk aversion in terms of constructs such as risk premia, certainty equivalents, Arrow-Pratt coefficients and acceptance sets. Given that these derived concepts are used in various applications, a natural question arises: can a risk averse decision-maker's preference, usually represented by a von Neumann-Morgenstern utility function, be represented equivalently in terms of the derived constructions? In other

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words, can the decision-maker's von Neumann-Morgenstern utility function be recovered from the decision-maker's risk premia, certainty equivalents, acceptance sets or Arrow-Pratt coefficients? Our objective in this paper is to identify a general class of preferences for which these questions can be answered in the affirmative. Towards this end, we shall establish dualities between the set of continuous and increasing risk averse von Neumann-Morgenstern utility functions and sets of multi-valued mappings that generate risk premia, certainty equivalents and acceptance sets, not just in the real outcomes case, but very generally in the vector outcomes case.

Consider a decision-making setting with an outcome space O that is a subset of a partially ordered vector space X. Let $\Delta(O)$ be the set of σ-additive probability measures (henceforth, referred to as lotteries) on O. Let $\mathcal{U}$ be a set of von Neumann-Morgenstern (henceforth, abbreviated to vN-M) utility functions $u : O \to \Re$. Let $\mathcal{F}$ be a set of mappings $F : \Delta(O) \Rightarrow O$, where $F(\mu)$ is interpreted as the set of certainty equivalent outcomes (Pratt [19]) corresponding to a lottery μ.[1] Let $\mathcal{P}$ be a set of mappings $P : \Delta(O) \Rightarrow X$, where $P(\mu)$ is interpreted as the set of risk premia (Pratt [19]) corresponding to a lottery μ. Unlike in the real outcomes setting, the notions of certainty equivalent outcomes and risk premia are necessarily set-valued in the vector outcomes setting. Finally, let $\mathcal{A}$ be a set of mappings $A : O \Rightarrow \Delta(O)$, where $A(x)$ is interpreted as the acceptance set (Yaari [22]) corresponding to an outcome x. We shall specify the sets $\mathcal{U}$, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{A}$ by imposing appropriate requirements on their elements; e.g., elements of $\mathcal{U}$ are continuous, weakly risk averse and order-preserving with respect to the given partial order on X. The principal contribution of this paper is to show the existence of bijections $\phi : \mathcal{U} \to \mathcal{F}$ (Theorem 3.6), $\psi : \mathcal{U} \to \mathcal{P}$ (Theorem 4.1) and $\xi : \mathcal{U} \to \mathcal{A}$ (Theorem 5.5); clearly, these bijections generate other bijections $\psi \circ \phi^{-1} : \mathcal{F} \to \mathcal{P}$, $\xi \circ \phi^{-1} : \mathcal{F} \to \mathcal{A}$ and $\xi \circ \psi^{-1} : \mathcal{P} \to \mathcal{A}$.

We describe our methodology with respect to the duality ϕ between $\mathcal{U}$ and $\mathcal{F}$; analogous descriptions apply to the other dualities too. Given $u \in \mathcal{U}$ and a lottery $\mu \in \Delta(O)$, the set of certainty equivalents $\phi(u)(\mu)$ is defined in the natural way and it is straightforward to confirm that the resulting mapping $\phi(u) : \Delta(O) \Rightarrow O$ satisfies the properties that define the elements of $\mathcal{F}$, i.e., $\phi(u) \in \mathcal{F}$. Next, we show that ϕ is injective. The final step is to show that ϕ is surjective. We show this as follows. Given $F \in \mathcal{F}$, we define a complete preordering $\succeq_F$ on $\Delta(O)$; let $\succ_F$ be the asymmetric factor of $\succeq_F$. The vN-M representation problem with respect to $\succ_F$ is to find $u_F : O \to \Re$ such that, for all $\mu, \lambda \in \Delta(O)$, $\mu \succ_F \lambda$ if and only if $\int_O \mu(dz)\, u_F(z) > \int_O \lambda(dz)\, u_F(z)$. We show that ϕ is surjective by showing that the vN-M representation problem with respect to $\succ_F$ has a solution

[1] We use $\Rightarrow$ to denote set-valued mappings as well as logical implication. The intended meaning should be clear from the context.

$u_F : O \to \Re$ such that $u_F \in \mathcal{U}$ and $\phi(u_F) = F$.

Our results imply that the information embedded in a mapping from one of the defined sets of mappings can also be recovered from one and only one mapping from each of the other sets of mappings as well. These results promise to ease applications of the theory of risk aversion, especially in the context of vector outcomes. This is because, in applications of the theory of risk aversion, one is usually concerned with objects such as risk premia, certainty equivalents and acceptance sets, while the vN-M utility function is merely the means for systematically generating these objects of substantive interest. Our results allow one to directly specify and work with the objects of interest, safe in the knowledge that, if these objects satisfy the properties we postulate, then they are indeed generated by some vN-M utility, and therefore are well-grounded in expected utility theory.

While the classical theory of risk aversion is couched in terms of real outcomes, which are usually interpreted as "wealth", there are many potential applications where outcomes are properly thought of as vectors. Financial economics is an area where the outcomes relevant to a decision-maker are typically vectors. In intertemporal financial models, decision-makers are routinely faced with the problem of choosing among assets whose returns are random processes. Since random processes can be represented by lotteries over a designated set of sample paths, the decision problem is essentially one of choosing among lotteries over sample paths. These sample paths are the relevant outcomes for the decision-maker and cannot be generally reduced to a scalar "wealth" outcome. This class of problems, exemplified by Application 6.4, also motivates the generality of our formalism. As sample paths in financial economics are typically continuous functions or belong to an even more general vector space, a useful theory should strive to specify X and O as generally as tractable and necessary.[2]

X being a general vector space instead of $X = \Re$ has a number of implications for the problem at hand. The salient ones are as follows. First, if $X = \Re$, then the usual order $>$ on $\Re$ is complete. On the other hand, with X a vector space, we shall only require that $>$ be a partial order on it; even in the simplest cases, we do not have complete orders that are tractable and economically interpretable. Secondly, if $X = \Re$ and the vN-M utility $u : O \to \Re$ is strictly increasing, then the sets of risk premia and certainty equivalents are singletons. If X is a vector space, then these constructs cease to be singleton-valued. An example of the problems created by this fact is the question: what meaning is to be ascribed to the relation "the risk premia generated by a lottery μ are larger than the risk premia

[2]For instance, the Wiener measure on the space of continuous sample paths results in the coordinate process being the Wiener process, which generates Brownian motion and geometric Brownian motion *via* elementary transformations. Itô and McKean [10] is the classic reference for the mathematics of diffusions and Duffie [3] is a useful introduction to the financial theory applications.

generated by lottery λ"? In the case $X = \Re$, as the sets of risk premia are singletons, this question is answered using the usual ordering on $\Re$. Thirdly, the vector outcome setting also forces one to address additional technical complications. For instance, we shall need to define the mean of a lottery over quite general vector outcomes, which entails integrating vector-valued functions. In this regard, it is important to confirm that a unique mean exists for every lottery.

Before plunging into the details, we briefly discuss the literature on risk aversion in the vector outcome setting. There are two distinct strands in this literature. One strand (Duncan [4], Karni [11], Kihlstrom and Mirman [12], [13], Levy and Levy [15], Shah [20]) studies risk aversion directly in the context of vector-valued risks that are given as primitive objects, as we do in this paper. The emphasis in these papers is to develop measures of risk aversion and notions of comparative risk aversion that are appropriate in the vector-outcome context. The other strand (Grant et al. [6], [7], Hanoch [8], Martinez-Legaz and Quah [16], Stiglitz [21]) studies the relationship between vector-valued risks (lotteries on commodity bundles) and real-valued risks (lotteries on wealth) when they are linked by a consumer's budget constraint. It is natural in this setting to interpret the von Neumann-Morgenstern utility function on a real domain as the indirect utility function for a fixed price vector. For each price vector, this enables the application of the classical theory of risk aversion couched in terms of real-valued outcomes. This context also permits a restricted indirect theory of choice among vector-valued risks since lotteries over wealth levels amount to lotteries over commodity bundles on the Engel curve corresponding to a given price vector. While Stiglitz [21] explores the implications of the purely indirect approach, Grant et al. ([6], [7]) and Martinez-Legaz and Quah [16] study the nature and extent of duality between the direct and indirect approaches.

The rest of this paper is organized as follows. In Section 2, we describe the formal requirements on X, O, $\Delta(O)$ and $\succeq$. We also define the mean m_μ of a lottery $\mu \in \Delta(O)$ and note that, given our formal context, every lottery $\mu \in \Delta(O)$ has a unique mean. In Section 3, we define the class of (weakly) risk averse utility functions $\mathcal{U}$ and the class $\mathcal{F}$ of mappings that generate lottery-contingent certainty equivalent outcomes. The analysis of this section leads up to the duality result in Theorem 3.6. In Section 4, we define the class $\mathcal{P}$ of lottery-contingent risk premia. The analysis of this section leads up to the duality result in Theorem 4.1. Finally, in Section 5, we define the class $\mathcal{A}$ of outcome-contingent acceptable lotteries. The analysis of this section leads up to the duality result in Theorem 5.5. Section 6 is devoted to applications of the duality results. We show in Theorem 6.1 that $F \in \mathcal{F}$ and $A \in \mathcal{A}$ are continuous mappings. We use these facts to compute the value of financial assets to a risk averse investor when the assets are characterized by a known or random stream of dividends. In Section 7, we compare the ordinal and cardinal utility representation problems. We summarize our results in Section 8.

2 Formal setting

Let X be a metrizable real locally convex topological vector space. Let $\geq$ be a reflexive, transitive and antisymmetric binary relation on X such that (a) if $x, y, z \in X$ and $x \geq y$, then $x+z \geq y+z$, and (b) if $x, y \in X$, $t \in \Re_{++}$ and $x \geq y$, then $tx \geq ty$. (a) requires invariance with respect to vector addition and (b) requires invariance with respect to scalar multiplication. Define the relation $>$ on X by: for $x, y \in X$, $x > y$ if and only if $x \geq y$ and $\neg y \geq x$. Let $X_+ = \{x \in X \mid x \geq 0\}$ be the positive cone of X. Given nonempty sets $E, F \subset X$, we say that $E \geq^* F$ if $\neg y > x$ for all $x \in E$ and $y \in F$.

Let O be a convex and compact subset of X_+ with $0 \in O$. Equip O with the subspace topology and the Borel σ-algebra $\mathcal{B}(O)$. As O is metrizable, every singleton subset of O is closed in O. Consequently, $\{x\} \in \mathcal{B}(O)$ for every $x \in O$.

Let $\Delta(O)$ be the set of σ-additive probability measures on O. Let $\mathcal{C}(O, \Re)$ denote the set of continuous functions $g : O \to \Re$. As O is compact, every $g \in \mathcal{C}(O, \Re)$ is bounded. Therefore, the formula $L(\mu, g) = \int_O \mu(dz)\, g(z)$ defines a linear functional $L(., g) : \Delta(O) \to \Re$ for every $g \in \mathcal{C}(O, \Re)$. We give $\Delta(O)$ the weak* topology, which is the coarsest topology on $\Delta(O)$ that makes every functional in $\{L(., g) \mid g \in \mathcal{C}(O, \Re)\}$ continuous, i.e., it is the projective topology generated on $\Delta(O)$ by the family $\{L(., g) \mid g \in \mathcal{C}(O, \Re)\}$.

We note some consequences of our assumptions. $\Delta(O)$ is compact and metrizable when given the weak* topology (Parthasarathy [17], Theorem II.6.4). As O is compact metric, it is separable, i.e., there is a countable set $E \subset O$ that is dense in O. Given $x \in O$, δ_x denotes the Dirac measure at x, i.e., for every $B \in \mathcal{B}(O)$, $\delta_x(B) = 1$ if $x \in B$ and $\delta_x(B) = 0$ otherwise. As $\{x\} \in \mathcal{B}(O)$ for every $x \in O$, $\delta_x \in \Delta(O)$ for every $x \in O$. Let $\Delta^0(E)$ denote the set of $\mu \in \Delta(O)$ with finite support in E, i.e., μ is a finite convex combination of Dirac measures in E. Then, $\Delta^0(E)$ is dense in $\Delta(O)$ (Parthasarathy [17], Theorem II.6.3). Given $\mu \in \Delta(O)$, $m_\mu = \int_O \mu(dz)\, z$ denotes the mean of μ, where the integral on the right-hand side is the Pettis integral; see Pettis [18] for details.

Theorem 2.1 *If O is nonempty, convex, compact and metrizable, and $\mu \in \Delta(O)$, then m_μ exists, is unique and $m_\mu \in O$.*

We say that $u : O \to \Re$ is weakly risk averse if $u(m_\mu) \geq \int_O \mu(dz)\, u(z)$ for every $\mu \in \Delta(O)$. For every function $u : O \to \Re$, the set of functions $[u] = \cup_{a \in \Re} \cup_{b \in \Re_{++}} \{v : O \to \Re \mid v = a + bu\}$ is an equivalence class. We shall formally identify $u : O \to \Re$ with the equivalence class $[u]$; a property α that holds for every $v \in [u]$ will be denoted simply by "u satisfies α". As indicated by our notion of equivalence classes, u is to be interpreted as a vN-M utility function.

3 Utility functions and certainty equivalents

Definition 3.1 *$\mathcal{U}$ is the set of functions $u : O \to \Re$ such that*
(a) u is continuous,
(b) u is weakly risk averse,
(c) u is increasing with respect to $>$, and
(d) $u(0) = 0$.

(a) is a regularity condition that is used for various existence arguments. (b) and (c) are salient and natural properties in most economic contexts. As $O \subset X_+$ and $u(0) = 0$ for $u \in \mathcal{U}$, (c) implies $u(x) \in \Re_+$ for every $x \in O$. However, (d) is not a substantive restriction on $\mathcal{U}$ in the sense that it does not restrict the class of preferences on $\Delta(O)$ that have a vN-M representation in $\mathcal{U}$. More precisely, if $u : O \to \Re$ satisfies properties (a) to (c), then $v : O \to \Re$ defined by $v = u - u(0)$ is equivalent to u and $v \in \mathcal{U}$. We now define a class of multi-valued mappings $F : \Delta(O) \Rightarrow O$ with the interpretation that $F(\mu)$ is the set of certainty equivalent outcomes corresponding to the lottery μ.

Definition 3.2 *$\mathcal{F}$ is the set of mappings $F : \Delta(O) \Rightarrow O$ such that*
(A) F has nonempty values,
(B) $\geq^*$ is a complete and antisymmetric preordering on $\{F(\mu) \mid \mu \in \Delta(O)\}$,
(C) for all $\mu, \lambda, \gamma \in \Delta(O)$, $F(\mu) = F(\lambda)$ implies $F(\mu/2+\gamma/2) = F(\lambda/2+\gamma/2)$,
(D) for every $\lambda \in \Delta(O)$, $\{\mu \in \Delta(O) \mid F(\mu) \geq^* F(\lambda)\}$ and $\{\mu \in \Delta(O) \mid F(\lambda) \geq^* F(\mu)\}$ are closed in $\Delta(O)$,
(E) $F(\delta_{m_\mu}) \geq^* F(\mu)$ for every $\mu \in \Delta(O)$,
(F) $x \in F(\delta_x)$ for every $x \in O$,
(G) if $x, y \in O$ and $x > y$, then $F(\delta_x) \geq^* F(\delta_y)$ and $\neg F(\delta_y) \geq^* F(\delta_x)$, and
(H) $x \in F(\mu)$ implies $F(\mu) = F(\delta_x)$.

For every $u \in \mathcal{U}$, define $\phi(u) : \Delta(O) \Rightarrow O$ by

$$\phi(u)(\mu) = \left\{ x \in O \,\middle|\, u(x) = \int_O \mu(dz)\, u(z) \right\}$$

Given a utility function u and a lottery μ, $\phi(u)(\mu)$ is the set of outcomes that yield the same utility as the expected utility derived from u and μ. In the case of scalar outcomes and an increasing utility function, the set of certainty equivalent outcomes is a singleton set; this is no longer the case when outcomes are vectors. Note that, if $u \in \mathcal{U}$ and $v = a + bu$ where $a \in \Re$ and $b \in \Re_{++}$, then $\phi(u) = \phi(v)$.

The main result of this section, Theorem 3.6, shows that ϕ is a bijection between $\mathcal{U}$ and $\mathcal{F}$. The proof is divided into three lemmas. In Lemma 3.3, we show that $\phi(u) \in \mathcal{F}$ for every $u \in \mathcal{U}$. In Lemma 3.4, we show that ϕ is injective. Finally, in Lemma 3.5, we show that ϕ is surjective by showing that $\phi^{-1}(\{F\}) \neq \emptyset$ for every $F \in \mathcal{F}$.

Lemma 3.3 *If $u \in \mathcal{U}$, then $\phi(u) \in \mathcal{F}$.*

Proof. Fix $u \in \mathcal{U}$. As u is fixed, denote $\phi(u)$ by F. (a) implies that u is measurable, and as O is compact, u is bounded. Therefore, the generalized Lebesgue integral $\int_O \mu(dz)\, u(z)$ exists for every $\mu \in \Delta(O)$. Define $U : \Delta(O) \to \Re$ by $U(\mu) = \int_O \mu(dz)\, u(z)$. Thus,

$$F(\mu) = \{x \in O \mid u(x) = U(\mu)\} = \{x \in O \mid U(\delta_x) = U(\mu)\} \tag{3.4}$$

As $\Delta(O)$ is given the weak* topology, (a) implies that U is continuous.

(A) As O is convex, it is connected. As O is nonempty and connected, (a) implies $u(O) \subset \Re$ is nonempty and connected. (b), (c) and (d) imply that $u(m_\mu) \geq U(\mu) \geq 0 = u(0)$. As $u(O)$ is connected, we have $U(\mu) \in [u(0), u(m_\mu)] \subset u(O)$. Consequently, there exists $x \in O$ such that $u(x) = U(\mu)$, i.e., $x \in F(\mu)$.

Before demonstrating the other properties of F, we note that

$$F(\mu) \geq^* F(\lambda) \quad \Leftrightarrow \quad U(\mu) \geq U(\lambda) \tag{3.5}$$

for all $\mu, \lambda \in \Delta(O)$.

Suppose $U(\mu) < U(\lambda)$. By (A), $F(\mu) \neq \emptyset$ and $F(\lambda) \neq \emptyset$. Let $x \in F(\mu)$ and $y \in F(\lambda)$. As $0 \in O$ and $O \subset X_+$, (c) and (d) imply $0 \leq u(x) = U(\mu) < U(\lambda) = u(y)$. By (d), $y > 0$. As O is convex and $0 \in O$, $ty \in O$ for every $t \in [0, 1)$. As $[0, 1)$ is connected and X is a topological vector space, $\{ty \mid t \in [0, 1)\}$ is connected. Then, (a) implies that $\{u(ty) \mid t \in [0, 1)\}$ is connected. (c) implies $\{u(ty) \mid t \in [0, 1)\} = [0, u(y))$. As $u(x) \in [0, u(y))$, there exists $t \in [0, 1)$ such that $u(ty) = u(x)$, i.e., $ty \in F(\mu)$. As $ty < y$, we have $\neg F(\mu) \geq^* F(\lambda)$.

Conversely, suppose $\mu, \lambda \in \Delta(O)$ and $\neg F(\mu) \geq^* F(\lambda)$. Then, there exists $x \in F(\mu)$ and $y \in F(\lambda)$ such that $y > x$. By (c), $u(y) > u(x)$. Thus, $U(\lambda) = u(y) > u(x) = U(\mu)$.

We now check that F satisfies (B) to (H).

(B) (3.5) implies that $\geq^*$ is a complete preordering on $\{F(\mu) \mid \mu \in \Delta(O)\}$. To see that $\geq^*$ is antisymmetric on $\{F(\mu) \mid \mu \in \Delta(O)\}$, suppose $\mu, \lambda \in \Delta(O)$ are such that $F(\mu) \geq^* F(\lambda)$ and $F(\lambda) \geq^* F(\mu)$. (3.5) implies that $U(\mu) = U(\lambda)$. It follows from (3.4) that $F(\mu) = F(\lambda)$.

(C) Suppose $\mu, \lambda, \gamma \in \Delta(O)$ and $F(\mu) = F(\lambda)$. As $\geq^*$ is reflexive on $\{F(\mu) \mid \mu \in \Delta(O)\}$, we have $F(\mu) \geq^* F(\lambda)$ and $F(\lambda) \geq^* F(\mu)$. By (3.5),

$U(\mu) = U(\lambda)$. The linearity of U implies $U(\mu/2+\gamma/2) = U(\mu)/2+U(\gamma)/2 = U(\lambda)/2 + U(\gamma)/2) = U(\lambda/2 + \gamma/2)$. By (3.5) and the antisymmetry of $\geq^*$ on $\{F(\mu) \mid \mu \in \Delta(O)\}$, we have $F(\mu/2 + \gamma/2) = F(\lambda/2 + \gamma/2)$.

(D) Consider $\lambda \in \Delta(O)$. By (3.5) and the continuity of U, $\{\mu \in \Delta(O) \mid F(\mu) \geq^* F(\lambda)\} = \{\mu \in \Delta(O) \mid U(\mu) \geq U(\lambda)\}$ is closed in $\Delta(O)$. Similarly, $\{\mu \in \Delta(O) \mid F(\lambda) \geq^* F(\mu)\}$ is closed in $\Delta(O)$.

(E) Suppose there exists $\mu \in \Delta(O)$ such that $\neg F(\delta_{m_\mu}) \geq^* F(\mu)$. Then, there exists $x \in F(\mu)$ and $y \in F(\delta_{m_\mu})$ such that $x > y$. By definition, $u(x) = U(\mu)$ and $u(y) = U(\delta_{m_\mu}) = u(m_\mu)$. As $x > y$, we have $u(m_\mu) = u(y) < u(x) = U(\mu)$, a contradiction of (b).

(F) For every $x \in O$, $u(x) = U(\delta_x)$, and so $x \in F(\delta_x)$.

(G) Consider $x, y \in O$ such that $x > y$. Let $x' \in F(\delta_x)$ and $y' \in F(\delta_y)$. If $y' > x'$, then (c) implies $u(y) = u(y') > u(x') = u(x)$, a contradiction. So, $F(\delta_x) \geq^* F(\delta_y)$. As $x \in F(\delta_x)$, $y \in F(\delta_y)$ and $x > y$, it follows that $\neg F(\delta_y) \geq^* F(\delta_x)$.

(H) Let $x \in F(\mu)$. If $y \in F(\delta_x)$, then $u(y) = U(\delta_x) = u(x) = U(\mu)$. So, $y \in F(\mu)$. Thus, $F(\delta_x) \subset F(\mu)$. If $y \in F(\mu)$, then $u(y) = U(\mu) = u(x) = U(\delta_x)$. So, $y \in F(\delta_x)$. Thus, $F(\mu) \subset F(\delta_x)$. ■

Clearly, a stronger version of (C) holds for $F = \phi(u)$: $F(\mu) = F(\lambda)$ implies $F(t\mu + (1-t)\gamma) = F(t\lambda + (1-t)\gamma)$ for every $t \in [0, 1]$. We require the weaker condition in order to make $\mathcal{F}$ as large a class of mappings as possible.

Lemma 3.4 *ϕ is injective.*

Proof. Consider $u, v \in \mathcal{U}$ such that $\phi(u) = \phi(v)$. We show that $[u] = [v]$.

(1) We first show that u and v induce the same ordering on O. Without loss of generality, suppose there exist $x, y \in O$ such that $u(x) \geq u(y)$ and $v(x) < v(y)$. Then there exists $t \in [0, 1)$ such that $ty \in O$ and $v(ty) = v(x)$. It follows that $ty \in \phi(v)(\delta_x)$. However, as (c) implies that $u(x) \geq u(y) > u(ty)$, we have $ty \notin \phi(u)(\delta_x)$, a contradiction.

(2) By (A), $\phi(u)$ and $\phi(v)$ have nonempty values. For $\nu \in \Delta(O)$, let $x_\nu \in \phi(u)(\nu)$. Given $\mu, \lambda \in \Delta(O)$, (1) implies

$$U(\mu) \geq U(\lambda) \;\Leftrightarrow\; u(x_\mu) \geq u(x_\lambda) \;\Leftrightarrow\; v(x_\mu) \geq v(x_\lambda) \;\Leftrightarrow\; V(\mu) \geq V(\lambda)$$

Thus, U and V are linear mappings on $\Delta(O)$ that induce the same ordering on $\Delta(O)$.

(3) As $\Delta(O)$ is compact and U is continuous, there exist $\alpha, \beta \in \Delta(O)$ such that $U(\alpha) \leq U(\mu) \leq U(\beta)$ for every $\mu \in \Delta(O)$. If $U(\alpha) = U(\beta)$, then U is constant over $\Delta(O)$, say $U(\mu) = k_U$ for every $\mu \in \Delta(O)$. If U is constant over $\Delta(O)$, then so is V. Let $V(\mu) = k_V$ for every $\mu \in \Delta(O)$. Setting $a = k_V - k_U$ and $b = 1$ implies $V = a + bU$.

Suppose $U(\beta) > U(\alpha)$. Then, $V(\beta) > V(\alpha)$. Define

$$a = V(\alpha) - U(\alpha)\left[\frac{V(\beta) - V(\alpha)}{U(\beta) - U(\alpha)}\right] \qquad \text{and} \qquad b = \frac{V(\beta) - V(\alpha)}{U(\beta) - U(\alpha)}$$

Note that $b > 0$. Now consider $\mu \in \Delta(O)$. We show that $V(\mu) = a + bU(\mu)$.

As $U(\mu) \in [U(\alpha), U(\beta)]$, there is a unique $t \in [0,1]$ such that $U(\mu) = tU(\beta) + (1-t)U(\alpha)$. As U is linear, $U(\mu) = U(t\beta + (1-t)\alpha)$. Therefore, $V(\mu) = V(t\beta + (1-t)\alpha)$ and

$$\begin{aligned} a + bU(\mu) &= V(\alpha) + b[U(\mu) - U(\alpha)] \\ &= V(\alpha) + bt[U(\beta) - U(\alpha)] \\ &= V(\alpha) + t[V(\beta) - V(\alpha)] \\ &= tV(\beta) + (1-t)V(\alpha) \\ &= V(t\beta + (1-t)\alpha) \\ &= V(\mu) \end{aligned}$$

It follows that $v(x) = V(\delta_x) = a + bU(\delta_x) = a + bu(x)$ for every $x \in O$. Thus, $[u] = [v]$. ■

Lemma 3.5 *If $F \in \mathcal{F}$, then $\phi^{-1}(\{F\}) \neq \emptyset$.*

Proof. Consider $F \in \mathcal{F}$. By (A), $F(\mu) \neq \emptyset$ for every $\mu \in \Delta(O)$. Define the relation $\succeq^*$ on $\Delta(O)$ by: $\mu \succeq^* \lambda$ if and only if $F(\mu) \geq^* F(\lambda)$. (B) implies that $\succeq^*$ is a complete preordering. Define the relation $\sim^*$ on $\Delta(O)$ by $\mu \sim^* \lambda$ if and only if $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$.

If $\mu, \lambda \in \Delta(O)$ are such that $\mu \sim^* \lambda$, then $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$. Therefore, $F(\mu) \geq^* F(\lambda)$ and $F(\lambda) \geq^* F(\mu)$. So, the antisymmetry property in (B) implies $F(\mu) = F(\lambda)$. Conversely, if $F(\mu) = F(\lambda)$, then the reflexivity property in (B) implies $F(\mu) \geq^* F(\lambda)$ and $F(\lambda) \geq^* F(\mu)$. Thus, $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$, and consequently, $\mu \sim^* \lambda$. Thus, $\mu \sim^* \lambda$ if and only if $F(\mu) = F(\lambda)$.

Consider $\mu, \lambda, \gamma \in \Delta(O)$ such that $\mu \sim^* \lambda$. Then, $F(\mu) = F(\lambda)$ and (C) implies that $F(\mu/2 + \gamma/2) = F(\lambda/2 + \gamma/2)$. Thus, $\mu/2 + \gamma/2 \sim^* \lambda/2 + \gamma/2$.

Given $\gamma \in \Delta(O)$, (D) implies that $S = \{\mu \in \Delta(O) \mid \mu \succeq^* \gamma\} = \{\mu \in \Delta(O) \mid F(\mu) \geq^* F(\gamma)\}$ is closed in $\Delta(O)$. Consider $\mu, \lambda, \gamma \in \Delta(O)$ and the function $f : [0,1] \to \Delta(O)$ defined by $f(t) = t\mu + (1-t)\lambda$. As f is continuous and S is closed in $\Delta(O)$,

$$\{t \in [0,1] \mid t\mu + (1-t)\lambda \succeq^* \gamma\} = \{t \in [0,1] \mid f(t) \succeq^* \gamma\} = f^{-1}(S)$$

is closed in $[0,1]$. By an analogous argument, $\{\mu \in \Delta(O) \mid \gamma \succeq^* \mu\} = \{\mu \in \Delta(O) \mid F(\gamma) \geq^* F(\mu)\}$ is closed in $\Delta(O)$ and $\{t \in [0,1] \mid \gamma \succeq^* t\mu + (1-t)\lambda\}$ is closed in $[0,1]$.

It follows (Herstein and Milnor [9], Theorem 8) that there exists a linear representation $V_F : \Delta(O) \to \Re$ of $\succeq^*$. Clearly, $U_F : \Delta(O) \to \Re$, defined by

$U_F(\mu) = V_F(\mu) - V_F(\delta_0)$, is a linear representation of $\succeq^*$ and $U_F(\delta_0) = 0$. As for every $\lambda \in \Delta(O)$, the sets $\{\mu \in \Delta(O) \mid U_F(\mu) \geq U_F(\lambda)\} = \{\mu \in \Delta(O) \mid \mu \succeq^* \lambda\}$ and $\{\mu \in \Delta(O) \mid U_F(\mu) \leq U_F(\lambda)\} = \{\mu \in \Delta(O) \mid \lambda \succeq^* \mu\}$ are closed in $\Delta(O)$, U_F is continuous. Define $u_F : O \to \Re$ by $u_F(z) = U_F(\delta_z)$. u_F is continuous (Parthasarathy [17], Theorem II.6.1) and $u_F(0) = U_F(\delta_0) = 0$.

Given $\mu \in \Delta(O)$, as $\Delta^0(E)$ is dense in $\Delta(O)$, there exists a sequence $(\mu_n) \subset \Delta^0(E)$ that converges to μ in the weak* topology. As each μ_n has finite support and U_F is linear,

$$\begin{aligned} U_F(\mu_n) &= U_F\left(\sum_{z \in \operatorname{supp}\mu_n} \mu_n(\{z\})\delta_z\right) \\ &= \sum_{z \in \operatorname{supp}\mu_n} \mu_n(\{z\}) U_F(\delta_z) \\ &= \sum_{z \in \operatorname{supp}\mu_n} \mu_n(\{z\}) u_F(z) \\ &= \int_O \mu_n(dz)\, u_F(z) \end{aligned}$$

Using the continuity of U_F and the definition of weak* convergence, we have

$$U_F(\mu) = \lim_{n\uparrow\infty} U_F(\mu_n) = \lim_{n\uparrow\infty} \int_O \mu_n(dz)\, u_F(z) = \int_O \mu(dz)\, u_F(z)$$

We now verify that $u_F \in \mathcal{U}$. By construction, u_F satisfies (a) and (d).

(b) Consider $\mu \in \Delta(O)$. By (E), $F(\delta_{m_\mu}) \geq^* F(\mu)$. So, $\delta_{m_\mu} \succeq^* \mu$. This implies $u_F(m_\mu) = U_F(\delta_{m_\mu}) \geq U_F(\mu) = \int_O \mu(dz)\, u_F(z)$.

(c) Consider $x, y \in O$ such that $x > y$. Then, (G) implies $F(\delta_x) \geq^* F(\delta_y)$ and $\neg F(\delta_y) \geq^* F(\delta_x)$. Consequently, $\delta_x \succeq^* \delta_y$ and $\neg \delta_y \succeq^* \delta_x$. Therefore, $u_F(x) = U_F(\delta_x) > U_F(\delta_y) = u_F(y)$.

Finally, we show that $u_F \in \phi^{-1}(\{F\})$, i.e., $\phi(u_F) = F$. We need to show that, for every $\mu \in \Delta(O)$,

$$F(\mu) = \{x \in O \mid u_F(x) = U_F(\mu)\} = \{x \in O \mid U_F(\delta_x) = U_F(\mu)\}$$

Observe that, for all $\mu, \lambda \in \Delta(O)$,

$$\begin{aligned} F(\mu) = F(\lambda) &\Leftrightarrow F(\mu) \geq^* F(\lambda) \ \wedge \ F(\lambda) \geq^* F(\mu) \\ &\Leftrightarrow \mu \succeq^* \lambda \ \wedge \ \lambda \succeq^* \mu \\ &\Leftrightarrow U_F(\mu) \geq U_F(\lambda) \ \wedge \ U_F(\lambda) \geq U_F(\mu) \\ &\Leftrightarrow U_F(\mu) = U_F(\lambda) \end{aligned}$$

The first equivalence follows from the fact that $\geq^*$ is reflexive and antisymmetric, while the second and third equivalences follow from the definitions of $\succeq^*$ and U_F.

Consider $x \in O$ such that $U_F(\delta_x) = U_F(\mu)$. It follows that $F(\delta_x) = F(\mu)$. By (F), $x \in F(\delta_x) = F(\mu)$. Conversely, consider $x \in F(\mu)$. By (H), $F(\mu) = F(\delta_x)$. Therefore, $U_F(\mu) = U_F(\delta_x)$. Thus, $\{x \in O \mid U_F(\delta_x) = U_F(\mu)\} = F(\mu)$. ■

As is evident from Theorem 8 in Herstein and Milnor [9] and the above proof, (D) is much stronger than the "continuity" condition that is sufficient for the existence of a linear representation of $\succeq^*$. However, the extra power of (D) is useful for showing that the derived linear representation is continuous and admits an expected utility representation.

Lemmas 3.3, 3.4 and 3.5 immediately yield

Theorem 3.6 *ϕ is a bijection from $\mathcal{U}$ to $\mathcal{F}$.*

4 Utility functions and risk premia

Another object of economic interest is the set of risk premia associated with a lottery. Given $u \in \mathcal{U}$, define $\psi(u) : \Delta(O) \rightrightarrows X$ by

$$\psi(u)(\mu) = \{y \in X \mid m_\mu - y \in O \ \wedge \ u(m_\mu - y) = U(\mu)\}$$

Given a utility function u and a lottery μ, a weakly risk averse decision-maker will prefer the mean m_μ of the lottery to the lottery itself, i.e., $u(m_\mu) \geq U(\mu)$. Given u and μ, $\psi(u)(\mu)$ is the set of all feasible non-random variations (risk premia) from the mean that leave the decision-maker indifferent between accepting the lottery and accepting the mean of the lottery adjusted by the risk premia. As in the case of certainty equivalents, while the set of risk premia is a singleton set when outcomes are scalars and u is increasing, this is not the case when outcomes are vectors.

Define $T : \Delta(O) \times X \to X$ by $T(\mu, x) = m_\mu - x$. As X is a topological vector space, given $\mu \in \Delta(O)$, $T(\mu, .)$ is a homeomorphism. Define

$$\mathcal{P} = \bigcup_{F \in \mathcal{F}} \{P : \Delta(O) \rightrightarrows X \mid P(.) = T(., F(.))\}$$

Clearly, $\beta : \mathcal{F} \to \mathcal{P}$, defined by $\beta(F)(.) = T(., F(.))$, is a bijection. It follows that $\psi = \beta \circ \phi$. Given $u \in \mathcal{U}$ and $\mu \in \Delta(O)$, we have the identities

$$\phi(u)(\mu) = T(\mu, \psi(u)(\mu)) \qquad \text{and} \qquad \psi(u)(\mu) = T(\mu, \phi(u)(\mu))$$

The following duality result follows.

Theorem 4.1 *ψ is a bijection from $\mathcal{U}$ to $\mathcal{P}$.*

5 Utility functions and acceptance sets

We now establish the duality between $\mathcal{U}$ and the class of mapping $A : O \Rightarrow \Delta(O)$ that yield the acceptance set $A(x) \subset \Delta(O)$ for every outcome $x \in O$. Given $A : O \Rightarrow \Delta(O)$, define the lower inverse mapping $A^- : \Delta(O) \Rightarrow O$ by $A^-(\mu) = \{x \in O \mid \mu \in A(x)\}$.

Definition 5.1 *$\mathcal{A}$ is the set of mappings $A : O \Rightarrow \Delta(O)$ such that*
(A) A^- has nonempty values,
(B) $\subset$ is a complete preordering on $\{A^-(\mu) \mid \mu \in \Delta(O)\}$,
(C) for every $\lambda \in \Delta(O)$, $\{\mu \in \Delta(O) \mid A^-(\mu) \subset A^-(\lambda)\}$ and $\{\mu \in \Delta(O) \mid A^-(\mu) \supset A^-(\lambda)\}$ are closed in $\Delta(O)$,
(D) for all $\mu, \lambda, \gamma \in \Delta(O)$, if $A^-(\lambda) = A^-(\mu)$, then $A^-(\lambda/2 + \gamma/2) = A^-(\mu/2 + \gamma/2)$,
(E) for $\mu \in \Delta(O)$ and $x \in O$, $x \in A^-(\mu)$ if and only if $A^-(\delta_x) \subset A^-(\mu)$,
(F) for every $x \in O$, $\mu \in A(x)$ implies $\neg x > m_\mu$, and
(G) for all $x, y \in O$, $x > y$ implies $A^-(\delta_x) \supset A^-(\delta_y)$ and $A^-(\delta_x) \not\subset A^-(\delta_y)$.

Given $u \in \mathcal{U}$, define $\xi(u) : O \Rightarrow \Delta(O)$ by

$$\xi(u)(x) = \left\{ \mu \in \Delta(O) \;\middle|\; u(x) \leq \int_O \mu(dz)\, u(z) \right\}$$

The main result of this section, Theorem 5.5, shows that ξ is a bijection between $\mathcal{U}$ and $\mathcal{A}$. The proof is divided into three lemmas. In Lemma 5.2, we show that $\xi(u) \in \mathcal{A}$ for every $u \in \mathcal{U}$. In Lemma 5.3, we show that ξ is injective. Finally, in Lemma 5.4, we show that ξ is surjective by showing that $\xi^{-1}(\{A\}) \neq \emptyset$ for every $A \in \mathcal{A}$.

Lemma 5.2 *If $u \in \mathcal{U}$, then $\xi(u) \in \mathcal{A}$.*

Proof. Fix $u \in \mathcal{U}$, replace $\xi(u)$ by A and define $U : \Delta(O) \to \Re$ by $U(\mu) = \int_O \mu(dz)\, u(z)$. Note that, for every $z \in O$ and $\mu \in \Delta(O)$,

$$z \in A^-(\mu) \qquad \Leftrightarrow \qquad \mu \in A(z) \qquad \Leftrightarrow \qquad U(\mu) \geq u(z)$$

It follows that, for all $\mu, \lambda \in \Delta(O)$,

$$\begin{aligned} A^-(\mu) \subset A^-(\lambda) &\Leftrightarrow [z \in A^-(\mu) \quad \Rightarrow \quad z \in A^-(\lambda)] && (1) \\ &\Leftrightarrow [U(\mu) \geq u(z) \quad \Rightarrow \quad U(\lambda) \geq u(z)] && (2) \\ &\Leftrightarrow U(\lambda) \geq U(\mu) && (3) \end{aligned}$$

(A) (c) and (d) imply that $u(x) \geq 0$ for every $x \in O$. Therefore, for every $\mu \in \Delta(O)$, we have $U(\mu) \geq 0 = u(0)$. It follows that $\mu \in A(0)$, i.e., $0 \in A^-(\mu)$.

(B) Reflexivity and transitivity of $\subset$ is trivial. Given $\mu, \lambda \in \Delta(O)$, we have either $U(\mu) \geq U(\lambda)$ or $U(\mu) \leq U(\lambda)$. Thus, either $A^-(\mu) \supset A^-(\lambda)$ or $A^-(\mu) \subset A^-(\lambda)$.

(C) (a) implies that U is continuous. Therefore, given $\lambda \in \Delta(O)$, $\{\mu \in \Delta(O) \mid A^-(\mu) \subset A^-(\lambda)\} = \{\mu \in \Delta(O) \mid U(\mu) \leq U(\lambda)\}$ is closed in $\Delta(O)$. Analogously, $\{\mu \in \Delta(O) \mid A^-(\mu) \supset A^-(\lambda)\}$ is closed in $\Delta(O)$.

(D) Consider $\mu, \lambda, \gamma \in \Delta(O)$ such that $A^-(\lambda) = A^-(\mu)$. (5.3) implies $U(\mu) = U(\lambda)$. It follows that $U(\mu/2+\gamma/2) = U(\mu)/2+U(\gamma)/2 = U(\lambda)/2+U(\gamma)/2 = U(\lambda/2+\gamma/2)$. (5.3) implies $A^-(\lambda/2+\gamma/2) = A^-(\mu/2+\gamma/2)$.

(E) Suppose $x \in A^-(\mu)$. Then, $\mu \in A(x)$, i.e., $U(\mu) \geq u(x) = U(\delta_x)$. If $z \in A^-(\delta_x)$, then $\delta_x \in A(z)$, i.e., $u(x) = U(\delta_x) \geq u(z)$. This implies $U(\mu) \geq u(z)$. So, $\mu \in A(z)$, i.e., $z \in A^-(\mu)$. Thus, $A^-(\delta_x) \subset A^-(\mu)$.

Conversely, suppose $A^-(\delta_x) \subset A^-(\mu)$, i.e., $z \in A^-(\delta_x)$ implies $z \in A^-(\mu)$. Equivalently, $\delta_x \in A(z)$ implies $\mu \in A(z)$. Therefore, $u(x) = U(\delta_x) \geq u(z)$ implies that $U(\mu) \geq u(z)$. Thus, $U(\mu) \geq u(x)$, which implies $\mu \in A(x)$, i.e., $x \in A^-(\mu)$.

(F) If $\mu \in A(x)$ and $x > m_\mu$, then (c) implies $U(\mu) \geq u(x) > u(m_\mu)$, which contradicts (b).

(G) Suppose $x > y$. Consider $z \in A^-(\delta_y)$. Then, $\delta_y \in A(z)$, i.e., $u(y) = U(\delta_y) \geq u(z)$. (c) implies $u(x) > u(y)$. Therefore, $u(x) > u(z)$. It follows that $U(\delta_x) = u(x) > u(z)$, i.e., $\delta_x \in A(z)$. This means $z \in A^-(\delta_x)$. Thus, $A^-(\delta_y) \subset A^-(\delta_x)$.

Suppose $A^-(\delta_x) \subset A^-(\delta_y)$. Then, $z \in A^-(\delta_x)$ implies $z \in A^-(\delta_y)$, i.e., $\delta_x \in A(z)$ implies $\delta_y \in A(z)$. Equivalently, $u(x) = U(\delta_x) \geq u(z)$ implies $u(y) = U(\delta_y) \geq u(z)$. This means $u(y) \geq u(x)$, a contradiction of (c). ■

We now show that $\xi : \mathcal{U} \to \mathcal{A}$ is an injection.

Lemma 5.3 *$\xi : \mathcal{U} \to \mathcal{A}$ is an injection.*

Proof. Consider $u, v \in \mathcal{U}$ such that $\xi(u) = \xi(v)$. Define the binary relations $\succeq_u^*$ and $\succeq_v^*$ on $\Delta(O)$ as follows: for all $\mu, \lambda \in \Delta(O)$, $\mu \succeq_u^* \lambda$ if and only if $U(\mu) \geq U(\lambda)$, and $\mu \succeq_v^* \lambda$ if and only if $V(\mu) \geq V(\lambda)$. Combining this with (5.3), we have

$$\mu \succeq_u^* \lambda \qquad \Leftrightarrow \qquad U(\mu) \geq U(\lambda) \qquad \Leftrightarrow \qquad \xi(u)^-(\lambda) \subset \xi(u)^-(\mu)$$

and

$$\mu \succeq_v^* \lambda \qquad \Leftrightarrow \qquad V(\mu) \geq V(\lambda) \qquad \Leftrightarrow \qquad \xi(v)^-(\lambda) \subset \xi(v)^-(\mu)$$

As $\xi(u) = \xi(v)$, $\mu \succeq_u^* \lambda$ if and only if $\mu \succeq_v^* \lambda$. It follows that

$$\begin{aligned} U(\mu) \geq U(\lambda) &\Leftrightarrow \mu \succeq_u^* \lambda \\ &\Leftrightarrow \mu \succeq_v^* \lambda \\ &\Leftrightarrow V(\mu) \geq V(\lambda) \end{aligned}$$

Copying the argument of Lemma 3.4, there exists $a \in \Re$ and $b \in \Re_{++}$ such that $V = a + bU$, i.e., $v = a + bu$. Thus, $[u] = [v]$. ■

Lemma 5.4 *If $A \in \mathcal{A}$, then $\xi^{-1}(\{A\}) \neq \emptyset$.*

Proof. Fix $A \in \mathcal{A}$. Define the relation $\succeq^*$ on $\Delta(O)$ by: $\mu \succeq^* \lambda$ if and only if $A^-(\mu) \supset A^-(\lambda)$. (B) implies that $\succeq^*$ is a complete preordering. Define the relation $\sim^*$ on $\Delta(O)$ by $\mu \sim^* \lambda$ if and only if $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$.

If $\mu, \lambda \in \Delta(O)$ are such that $\mu \sim^* \lambda$, then $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$. Therefore, $A^-(\mu) \supset A^-(\lambda)$ and $A^-(\mu) \subset A^-(\lambda)$. It follows that $A^-(\mu) = A^-(\lambda)$. Conversely, if $A^-(\mu) = A^-(\lambda)$, then $A^-(\mu) \supset A^-(\lambda)$ and $A^-(\mu) \subset A^-(\lambda)$. Thus, $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$, and consequently, $\mu \sim^* \lambda$. Thus, $\mu \sim^* \lambda$ if and only if $A^-(\mu) = A^-(\lambda)$.

Consider $\mu, \lambda, \gamma \in \Delta(O)$ such that $\mu \sim^* \lambda$. Then, $A^-(\mu) = A^-(\lambda)$, and (D) implies that $A^-(\mu/2 + \gamma/2) = A^-(\lambda/2 + \gamma/2)$. Thus, $\mu/2 + \gamma/2 \sim^* \lambda/2 + \gamma/2$.

Given $\gamma \in \Delta(O)$, (C) implies that $S = \{\mu \in \Delta(O) \mid \mu \succeq^* \gamma\} = \{\mu \in \Delta(O) \mid A^-(\mu) \supset A^-(\gamma)\}$ is closed in $\Delta(O)$. Consider $\mu, \lambda, \gamma \in \Delta(O)$ and the function $f : [0,1] \to \Delta(O)$ defined by $f(t) = t\mu + (1-t)\lambda$. As f is continuous and S is closed in $\Delta(O)$,

$$\{t \in [0,1] \mid t\mu + (1-t)\lambda \succeq^* \gamma\} = \{t \in [0,1] \mid f(t) \succeq^* \gamma\} = f^{-1}(S)$$

is closed in $[0,1]$. By an analogous argument, $\{\mu \in \Delta(O) \mid \gamma \succeq^* \mu\} = \{\mu \in \Delta(O) \mid A^-(\mu) \subset A^-(\gamma)\}$ is closed in $\Delta(O)$ and $\{t \in [0,1] \mid \gamma \succeq^* t\mu + (1-t)\lambda\}$ is closed in $[0,1]$.

It follows (Herstein and Milnor [9], Theorem 8) that $\succeq^*$ has a linear representation $V_A : \Delta(O) \to \Re$. Define $U_A : \Delta(O) \to \Re$ by $U_A(\mu) = V_A(\mu) - V_A(\delta_0)$. Clearly, U_A is a linear representation of $\succeq^*$ and $U_A(\delta_0) = 0$. Define $u_A : O \to \Re$ by $u_A(x) = U_A(\delta_x)$. Copying the argument of Lemma 3.5, U_A and u_A are continuous, and for every $\mu \in \Delta(O)$, $U_A(\mu) = \int_O \mu(dz)\, u_A(z)$. By definition, u_A satisfies (a) and (d).

(b) Suppose there exists $\mu \in \Delta(O)$ such that $U_A(\mu) > u_A(m_\mu)$. As O is convex, it is connected. As u_A is continuous and O is compact and connected, $u_A(O)$ is a closed interval in $\Re$. Thus, $U_A(\mu) \in u_A(O)$, i.e., there exists $x \in O$ such that $u_A(m_\mu) < U_A(\mu) = u_A(x)$. By the continuity of u_A, there exists $y > m_\mu$ such that $u_A(m_\mu) < u_A(y) < u_A(x) = U_A(\mu)$, i.e., $U_A(\delta_y) < U_A(\mu)$. As U_A represents $\succeq^*$, this means $A^-(\delta_y) \subset A^-(\mu)$. By (E), we have $y \in A^-(\mu)$, i.e., $\mu \in A(y)$. As $y > m_\mu$ we have a contradiction of (F).

(c) Suppose $x > y$. By (G), $A^-(\delta_x) \supset A^-(\delta_y)$ and $A^-(\delta_x) \not\subset A^-(\delta_y)$. Then, $\delta_x \succeq^* \delta_y$ and $\neg\delta_y \succeq^* \delta_x$. It follows that $u_A(x) = U_A(\delta_x) > U_A(\delta_y) = u_A(y)$.

Finally, we show that $u_A \in \xi^{-1}(\{A\})$, i.e., $\xi(u_A) = A$. We need to show that, for every $x \in O$,

$$A(x) = \{\mu \in \Delta(O) \mid u_A(x) \leq U_A(\mu)\} = \{\mu \in \Delta(O) \mid U_A(\delta_x) \leq U_A(\mu)\}$$

Fix $x \in O$. As, for all $\mu, \lambda \in \Delta(O)$,

$$U_A(\mu) \geq U_A(\lambda) \qquad \Leftrightarrow \qquad \mu \succeq^* \lambda \qquad \Leftrightarrow \qquad A^-(\lambda) \subset A^-(\mu)$$

the problem reduces to showing that

$$A(x) = \{\mu \in \Delta(O) \mid A^-(\delta_x) \subset A^-(\mu)\}$$

Using the definition of A^- and (E), we have

$$\mu \in A(x) \qquad \Leftrightarrow \qquad x \in A^-(\mu) \qquad \Leftrightarrow \qquad A^-(\delta_x) \subset A^-(\mu)$$

as required. ■

As is evident from Theorem 8 in Herstein and Milnor [9] and the above proof, (C) is much stronger than the "continuity" condition that is sufficient for the existence of a linear representation of $\succeq^*$. However, the extra power of (C) is useful for showing that the derived linear representation is continuous and admits an expected utility representation.

Combining Lemmas 5.2, 5.3 and 5.4, we have

Theorem 5.5 *$\xi : \mathcal{U} \to \mathcal{A}$ is a bijection.*

6 Applications

Duality results have two competing aspects. On the one hand, when choosing a dual representation such as $F \in \mathcal{F}$ or $A \in \mathcal{A}$ to specify a preference, we want the properties defining $\mathcal{F}$ or $\mathcal{A}$ to be minimal so that the modeller has greater latitude in selecting an appropriate F or A. On the other hand, when using the chosen dual representation, the modeller is free to use not only the properties used to define $\mathcal{F}$ or $\mathcal{A}$, but also other stronger properties possessed by elements of $\mathcal{F}$ or $\mathcal{A}$. So, an important aspect of duality theory is to derive various non-definitional properties possessed by dual representations. Our first application of the above duality results will derive such non-definitional properties of $\mathcal{F}$ and $\mathcal{A}$. The proofs of the following three results are relegated to the Appendix.

Theorem 6.1 *If $u \in \mathcal{U}$, then*
(A) $\phi(u)$ is continuous,
(B) $\xi(u)$ and $\xi(u)^-$ are continuous.
Moreover, every $F \in \mathcal{F}$ and $A \in \mathcal{A}$ is continuous.

Combining this result with Berge's Maximum theorem yields the following useful facts.

Theorem 6.2 *If $P : O \to \Re$ is continuous and $F \in \mathcal{F}$, then the mapping $V : \Delta(O) \to \Re$ defined by $V(\mu) = \min\{P(x) \mid x \in F(\mu)\}$ is continuous and the mapping $M : \Delta(O) \Rightarrow O$ defined by $M(\mu) = \{x \in F(\mu) \mid P(x) = V(\mu)\}$ is upper hemicontinuous with nonempty and compact values.*

A dual result is the following.

Theorem 6.3 *If $p : O \to \Re$ is continuous and $A \in \mathcal{A}$, then $v : O \to \Re$ defined by $v(x) = \min\{\int_O \mu(dz)\, p(z) \mid \mu \in A(x)\}$ is continuous; moreover, $B : O \Rightarrow \Delta(O)$ defined by $B(x) = \{\mu \in A(x) \mid \int_O \mu(dz)\, p(z) = v(x)\}$ is upper hemicontinuous with nonempty and compact values.*

As an application of Theorems 6.2 and 6.3, consider the following problem.

Application 6.4 *Let $\{1, \ldots, n\}$ be the set of future dates. Let $X = \Re^n$, give X the Euclidean topology and let $O \subset X_+$ be convex and compact with $0 \in O$. $x \in O$ is interpreted as an asset dividend path, with x_t being the dividend paid at date $t \in \{1, \ldots, n\}$. An asset is denoted by $\mu \in \Delta(O)$. Asset μ is said to be riskless if $\mu = \delta_x$ for some $x \in O$, and risky otherwise. Let asset prices be given by $P : \Delta(O) \to \Re$, where $P(\mu)$ is the price of asset μ and P is continuous when $\Delta(O)$ is given the weak* topology. A portfolio of assets is a function $\theta : \Delta(O) \to \Re$ with finite support, i.e., $|\mathrm{supp}\, \theta| \equiv |\overline{\theta^{-1}(\Re - \{0\})}| < \infty$. Suppose P is arbitrage-free, meaning that there is no portfolio θ of assets such that $\sum_{\mu \in \mathrm{supp}\, \theta} \theta(\mu) P(\mu) < 0$ and $\sum_{\mu \in \mathrm{supp}\, \theta} \theta(\mu) \int_O \mu(dz)\, z \geq 0$, i.e., a portfolio with a negative acquisition cost and non-negative expected dividends. Given this set-up, what is the value to a risk averse investor of asset $\mu \in \Delta(O)$? How does this value vary with μ?*

If P permits an arbitrage in the above sense, then there exists a portfolio θ such that $\sum_{\mu \in \mathrm{supp}\, \theta} \theta(\mu) P(\mu) < 0$ and $\sum_{\mu \in \mathrm{supp}\, \theta} \theta(\mu) \int_O \mu(dz)\, z \geq 0$, i.e., a risk neutral investor would like to acquire an unboundedly large portfolio. Assuming the existence of a risk neutral investor, the above notion of "arbitrage-free" asset prices is a necessary property of equilibrium prices. The functional $\pi : O \to \Re$, defined by $\pi(x) = P(\delta_x)$, yields the prices of riskless assets. As O is separable, $\Delta^0(O)$ is dense in $\Delta(O)$ (Parthasarathy [17], Theorem II.6.3). We note some useful facts about P and π.

Lemma 6.5 *Consider Application 6.4.*

(A) π is linear on O, $\pi(0) = 0$ and $\pi(x) \geq 0$ for every $x \in O$.

(B) π is continuous.

(C) If every unit vector $e_t \in O$, then there exists $(\pi_1, \ldots, \pi_n) \in \Re^n_+$ such that $\pi(x) = \sum_{t=1}^n \pi_t x_t$.

(D) For every $\mu \in \Delta(O)$, $P(\mu) = \pi(m_\mu)$.

Proof. (A) follows from the fact that P is arbitrage-free. (B) follows as π is linear. (C) follows by setting $\pi_t = \pi(e_t)$.

(D) As P is arbitrage-free, if $\mu \in \Delta^0(O)$, then

$$P(\mu) = \sum_{z \in \operatorname{supp} \mu} \mu(\{z\})\pi(z) = \int_O \mu(dz)\,\pi(z) = \pi(m_\mu)$$

Consider $\mu \in \Delta(O)$. As $\Delta^0(O)$ is dense in $\Delta(O)$, there exists a net $(\mu_n) \subset \Delta^0(O)$ converging to μ. As P and π are continuous, we have $P(\mu) = \lim_n P(\mu_n) = \lim_n \int_O \mu_n(dz)\,\pi(z) = \int_O \mu(dz)\,\pi(z)$. As π is linear, we have $\int_O \mu(dz)\,\pi(z) = \pi(\int_O \mu(dz)\,z) = \pi(m_\mu)$. Thus, $P(\mu) = \pi(m_\mu)$. ■

We define an asset's value to an investor as the maximum amount that the investor would be willing to pay for it. By Theorems 3.6 and 5.5, a risk averse investor's preference on $\Delta(O)$ can be specified equivalently by $u \in \mathcal{U}$, $F \equiv \phi(u) \in \mathcal{F}$, or $A \equiv \xi(u) \in \mathcal{A}$. If the preference is represented by $u \in \mathcal{U}$, then the value of asset $\mu \in \Delta(O)$ to the given investor is $V(\mu) = \min\{P(\lambda) \mid \lambda \in \Delta(O) \,\wedge\, U(\lambda) \geq U(\mu)\}$. The next result provides dual characterizations of V and notes some properties.

Theorem 6.6 *Consider Application 6.4. Let $u \in \mathcal{U}$, $F \equiv \phi(u) \in \mathcal{F}$ and $A \equiv \xi(u) \in \mathcal{A}$.*

(A) If $\mu \in \Delta(O)$, then $V(\mu) = \min \pi \circ F(\mu)$.

(B) If $x \in O$, then $V(\delta_x) = \min \pi \circ F(\delta_x) = \min P \circ A(x)$.

(C) $P(\mu) \geq V(\mu)$ for every $\mu \in \Delta(O)$.

(D) V is continuous, and the mappings $\mu \rightrightarrows \{x \in O \mid \pi(x) = V(\mu)\}$ and $x \rightrightarrows \{\lambda \in \Delta(O) \mid P(\lambda) = V(\delta_x)\}$ are upper hemicontinuous with nonempty and compact values.

Proof. (A) If $x \in F(\mu)$, then $\delta_x \in \Delta(O)$ and $U(\delta_x) = u(x) = U(\mu)$. By definition, $V(\mu) \leq P(\delta_x) = \pi(x)$. It follows that $V(\mu) \leq \min \pi \circ F(\mu)$. Let $V(\mu) = P(\lambda)$ for some $\lambda \in \Delta(O)$ such that $U(\lambda) \geq U(\mu)$. As $u \in \mathcal{U}$, $u(m_\lambda) \geq U(\lambda) \geq U(\mu) \geq 0$. Consequently, there exists $t \in [0,1]$ such that $tm_\lambda \in F(\mu)$. As $m_\lambda \in O$, Lemma 6.5(A) implies that $\pi(m_\lambda) \geq 0$ and $\pi(tm_\lambda) = t\pi(m_\lambda) \leq \pi(m_\lambda)$. It follows that $\min \pi \circ F(\mu) \leq \pi(tm_\lambda) \leq \pi(m_\lambda) = P(\lambda) = V(\mu)$. Thus, $V(\mu) = \min \pi \circ F(\mu)$.

(B) Specializing (A), we have $\min \pi \circ F(\delta_x) = V(\delta_x) = \min\{P(\lambda) \mid \lambda \in \Delta(O) \,\wedge\, U(\lambda) \geq U(\delta_x)\} = \min P \circ A(x)$.

(C) If $\mu \in \Delta(O)$ is such that $P(\mu) < V(\mu)$, then $\pi(m_\mu) = P(\mu) < V(\mu) = \min \pi \circ F(\mu) \leq \pi(m_\mu)$, a contradiction.

(D) follows from (A), (B) and Theorems 6.2 and 6.3. ■

Now consider the following continuous-time analogue of Application 6.4. Let $[0,1]$ be the set of dates and let $X = \mathcal{C}([0,1],\Re)$ be the set of continuous real-valued functions with domain $[0,1]$. Let P and π be as in Application 6.4. Parts (A) and (D) of Lemma6.5 hold in this setting *via* unchanged arguments.

Lemma 6.5(B) is now proved as follows. Consider a sequence $(x_n) \subset O$ converging to x. If $f : O \to \Re$ is continuous, then $\int_O \delta_{x_n}(dz)\, f(z) = f(x_n) \to f(x) = \int_O \delta_x(dz)\, f(z)$. So, $(\delta_{x_n}) \subset \Delta(O)$ converges to δ_x. As P is continuous, $\pi(x_n) = P(\delta_{x_n}) \to P(\delta_x) = \pi(x)$. Thus, π is continuous.

The analogue of Lemma 6.5(C) is established as follows. By the Riesz representation theorem (Dunford and Schwartz [5], Theorem IV.6.3), there exists a unique, non-negative, regular σ-additive measure Q on $[0, 1]$ such that $\pi(x) = \int_{[0,1]} Q(dt)\, x(t)$ for every $x \in X$. As π is real-valued, Q is finite. If Q is absolutely continuous with respect to the Lebesgue measure on $[0, 1]$, then by the Radon-Nikodym theorem (Dunford and Schwartz [5], Theorem III.10.2), there exists a unique (upto equivalence) Lebesgue integrable function $q : [0, 1] \to \Re$ such that $Q(E) = \int_E dt\, q(t)$ for every $E \in \mathcal{B}([0, 1])$. Therefore, $\pi(x) = \int_{[0,1]} dt\, q(t)x(t)$ for every $x \in X$. As Q is non-negative, q is non-negative on $[0, 1]$, except possibly over a set of Lebesgue measure 0. As in Application 6.4, $q(t)$ is interpreted as the price of delivering \$1 at time t.

7 Ordinal and cardinal representation problems

The question motivating this paper is analogous to that motivating the familiar duality results in microeconomic theory, e.g., the dualities between direct utility functions, indirect utility functions and expenditure functions in consumer theory. Although our dual characterizations of risk averse vN-M utility functions and the dual characterizations of ordinal utility functions (e.g., Krishna and Sonnenschein [14]) are almost entirely different in aims, techniques and the objects being studied, there remains one seemingly common element. This common element is the set of functions $\mathcal{U}$ since a function $u : O \to \Re$ can be interpreted as an ordinal utility function or as a vN-M utility function. We analyze this formal commonality by noting three points of comparison between the theories.

The first observation relates to the quotient sets of $\mathcal{U}$ generated by the ordinal and the vN-M interpretations of the elements of $\mathcal{U}$.[3] If the elements of $\mathcal{U}$ are interpreted as ordinal utility functions, then elements $u, v \in \mathcal{U}$ are considered to be equivalent, denoted by $u \equiv^1 v$, if they are increasing transforms of each other. This notion of equivalence generates the quotient set $\mathcal{U}/\equiv^1$. On the other hand, if the elements of $\mathcal{U}$ are interpreted as vN-M utility functions, then elements $u, v \in \mathcal{U}$ are considered to be equivalent, denoted by $u \equiv^2 v$, if they are increasing affine transforms of each other. This notion of equivalence generates the quotient set $\mathcal{U}/\equiv^2$. It is easy to see that $\mathcal{U}/\equiv^2$ is a sub-partition of $\mathcal{U}/\equiv^1$, i.e., if $[u]_1 \in \mathcal{U}/\equiv^1$ and $[u]_2 \in \mathcal{U}/\equiv^2$ are the equivalence classes to which $u \in \mathcal{U}$ belongs, then

[3] The quotient set of a set S with respect to an equivalence relation $\equiv$ on S, denoted by $S/\equiv$, refers to the partition of S generated by $\equiv$.

$[u]_2 \subset [u]_1$ and $[u]_2 \neq [u]_1$.

The second observation relates to the nature of the surjection argument in the two theories. In the case of ordinal utility functions, the existence of a utility function generating a given expenditure function or indirect utility function is shown by verifying that explicitly displayed solutions of a constrained optimization problem yield the values of the required function (Krishna and Sonnenschein [14]). In our surjection proofs, as we cannot provide such explicit characterizations, we rely on the general existence results relating to linear utility functions contained in Herstein and Milnor [9].

The third observation relates to the representation problems underlying the two theories. Let $\succeq^*$ be a complete preordering on $\Delta(O)$ and let $\succeq$ be induced on O *via* the definition: $x \succeq y$ if and only if $\delta_x \succeq^* \delta_y$; let $\sim$ be the symmetric factor (indifference) of $\succeq$ and $\succ$ the asymmetric factor (strict preference). Define the function $o : O \to O/\sim$ by the formula: given $x \in O$, $z \in o(x)$ if and only if $z \sim x$; $o(x) \in O/\sim$ is the indifference curve containing x. If $o_1, o_2 \in O/\sim$ and $o_1 \neq o_2$, then either $x \succ y$ for all $x \in o_1$ and $y \in o_2$, or $y \succ x$ for all $x \in o_1$ and $y \in o_2$. Therefore, by identifying equivalent elements of O with the equivalence class to which they belong, we may say that $\succ$ orders the elements of $O/\sim$. The ordinal representation problem is to find $u : O/\sim \to \Re$ such that, for all $o_1, o_2 \in O/\sim$, $o_1 \succ o_2$ if and only if $u(o_1) > u(o_2)$, i.e., real numbers are assigned to the indifference curves in $O/\sim$ in a manner consistent with $\succ$. Subject to this ordinal requirement on the chosen real numbers, each indifference curve may be assigned a number in isolation from the numbers assigned to the other indifference curves. The vN-M representation problem is to find $u : O/\sim \to \Re$ such that, for all $\mu, \lambda \in \Delta(O)$, $\mu \succ^* \lambda$ if and only if $\int_O \mu(dz)\, u \circ o(z) > \int_O \lambda(dz)\, u \circ o(z)$. A solution u of this problem also solves the ordinal representation problem since

$$\begin{aligned} x \succ y &\Leftrightarrow \delta_x \succ^* \delta_y \\ &\Leftrightarrow \int_O \delta_x(dz)\, u \circ o(z) > \int_O \delta_y(dz)\, u \circ o(z) \\ &\Leftrightarrow u \circ o(x) > u \circ o(y) \end{aligned}$$

Unlike in the ordinal representation problem, the indifference curves in $O/\sim$ cannot be assigned values in isolation as the expected utility function aggregates these numbers *via* integration. Thus, the assigned values have cardinal, and not merely ordinal, significance. While the ordinal representation problem can be solved locally with respect to $O/\sim$, the vN-M representation problem has to be solved globally.

8 Conclusions

We have considered an outcome space O that is a convex and compact subset of the positive cone of a metrizable, partially ordered, real locally convex topological vector space X with $0 \in O$. Given this outcome space, we defined a class $\mathcal{U}$ of risk averse vN-M utility functions defined on O, a class $\mathcal{F}$ of multi-valued mappings that yield the certainty equivalent outcomes in O corresponding to a lottery in $\Delta(O)$, a class $\mathcal{P}$ of multi-valued mappings that yield the risk premia in X corresponding to a lottery in $\Delta(O)$, and a class $\mathcal{A}$ of multi-valued mappings that yield the acceptance set of lotteries in $\Delta(O)$ corresponding to an outcome in O.

We show that the usual definitions of the set of certainty equivalents, the set of risk premia and the acceptance set generate mappings $\phi : \mathcal{U} \to \mathcal{F}$, $\psi : \mathcal{U} \to \mathcal{P}$ and $\xi : \mathcal{U} \to \mathcal{A}$ respectively. Our main results (Theorems 3.6, 4.1 and 5.3) are that these mappings are bijective.

We also note in Theorem 6.1 that $\phi(u)$, $\xi(u)^-$ and $\xi(u)$ are continuous mappings for every $u \in \mathcal{U}$. Consequently, every $F \in \mathcal{F}$ and every $A \in \mathcal{A}$ is continuous. We use these facts to study two applications. Both applications involve the derivation of a risk averse investor's valuation of assets that are characterized by known or risky dividend paths. The first application derives such an investor's valuation of a risky asset and the second application derives the investor's valuation of a riskless asset. We reduce these problems to optimization problems and use our results to show that the value functions generated by these problems are continuous and the underlying optimal choice mappings are upper hemicontinuous.

Appendix

Proof of Theorem 2.1. Let X^* be the set of all continuous linear functionals $h : X \to \Re$. Local convexity of X ensures that, if $x \in X$ is such that $h(x) = 0$ for every $h \in X^*$, then $x = 0$ (Dunford and Schwartz [5], Corollary V.2.13). Define $H : X \to \Re^{X^*}$ by $H(x) = (h(x))_{h \in X^*}$. Give $\Re^{X^*}$ the product topology. Consequently, H is continuous as every component function $H_h = h$ is continuous. Moreover, H is injective; if $H(x) = H(y)$ for some $x, y \in X$, then $h(x - y) = h(x) - h(y) = 0$ for every $h \in X^*$, which implies $x - y = 0$. As O is compact and $\Re^{X^*}$ is Hausdorff, H imbeds O in $\Re^{X^*}$. This implies $H(O)$ is closed in $\Re^{X^*}$ and metrizable.

First, consider $\mu \in \Delta(O)$ with $|\operatorname{supp} \mu| < \infty$. For every $h \in H$, the linearity of h implies

$$\int_O \mu(dz)\, h(z) = \sum_{z \in \operatorname{supp} \mu} \mu(\{z\}) h(z) = h\left(\sum_{z \in \operatorname{supp} \mu} \mu(\{z\}) z \right) \qquad (A.1)$$

Setting $m_\mu = \sum_{z \in \operatorname{supp} \mu} \mu(\{z\}) z$, we have $m_\mu \in O$ as O is convex and $\operatorname{supp} \mu \subset O$. Thus, $H(m_\mu) \in H(O)$ for every $\mu \in \Delta(O)$ with $|\operatorname{supp} \mu| < \infty$.

Consider $\mu \in \Delta(O)$. As O is compact and metric, it is separable. Consequently, there exists a sequence $(\mu_n) \subset \Delta(O)$ converging to μ such that $|\operatorname{supp} \mu_n| < \infty$ for every $n \in \mathcal{N}$ (Parthasarathy [17], Theorem II.6.3). By the above argument, m_{μ_n} exists, $m_{\mu_n} \in O$ and $H(m_{\mu_n}) \in H(O)$ for every $n \in \mathcal{N}$. Using (A.1) and the definition of weak* convergence, we have

$$\lim_{n\uparrow\infty} h(m_{\mu_n}) = \lim_{n\uparrow\infty} \int_O \mu_n(dz)\, h(z) = \int_O \mu(dz)\, h(z)$$

for every $h \in X^*$. Thus, $\lim_{n\uparrow\infty} H(m_{\mu_n}) = (\int_O \mu(dz)\, h(z))_{h\in X^*}$. As the sequence $(H(m_{\mu_n})) \subset H(O)$ and $H(O)$ is closed in $\Re^{X^*}$ and metrizable, we have $(\int_O \mu(dz)\, h(z))_{h\in X^*} \in H(O)$. As H imbeds O in $\Re^{X^*}$, there exists a unique $x \in O$ such that $H(x) = (\int_O \mu(dz)\, h(z))_{h\in X^*}$. By the definition of H, we have $h(x) = \int_O \mu(dz)\, h(z)$ for every $h \in X^*$. Set $m_\mu = x$. ■

Proof of Theorem 6.1. Suppose (A) and (B) hold. Consider $F \in \mathcal{F}$ and $A \in \mathcal{A}$. By Theorem 3.6, $\phi^{-1}(F) \in \mathcal{U}$. Therefore, $F = \phi \circ \phi^{-1}(F)$ is continuous. By an analogous argument, A is continuous. It remains to prove (A) and (B).

Fix $u \in \mathcal{U}$, denote $\phi(u)$ by F, denote $\xi(u)$ by A, and denote the mapping $\mu \mapsto \int_O \mu(dz)\, u(z)$ by U. As $u \in \mathcal{U}$, u is continuous. As $\Delta(O)$ is given the weak* topology, U is continuous. Therefore, $G : \Delta(O) \times O \to \Re$, defined by $G(\mu, x) = U(\mu) - u(x)$, is continuous.

(A) It follows that $\operatorname{Gr} F = \{(\mu, x) \in \Delta(O) \times O \mid x \in F(\mu)\} = \{(\mu, x) \in \Delta(O) \times O \mid G(\mu, x) = 0\} = G^{-1}(\{0\})$ is closed in $\Delta(O) \times O$. As O is compact, F is upper hemicontinuous.

To show that F is lower hemicontinuous at $\mu \in \Delta(O)$, consider a sequence $(\mu_n) \subset \Delta(O)$ converging to μ and let $x \in F(\mu)$. As O and $\Delta(O)$ are metrizable, it is sufficient to construct a sequence $(x_n) \subset O$ converging to x such that $x_n \in F(\mu_n)$ for every $n \in \mathcal{N}$.

As U is continuous and $\Delta(O)$ is compact, there exists $\nu \in \Delta(O)$ such that $U(\nu) \geq U(\mu)$ for every $\mu \in \Delta(O)$. If $U(\nu) = 0$, then $O = \{0\}$ and lower hemicontinuity is trivial. Suppose $U(\nu) > 0$. As $F(\nu) \neq \emptyset$, there exists $y \in O$ such that $u(y) = U(\nu) > 0$. Thus, $y > 0$. We consider three cases.

(1) Suppose $U(\mu) = 0$. Then, $u(x) = 0$, i.e., $y > 0 = x$ and $U(\mu_n) \geq 0 = U(\mu)$ for every $n \in \mathcal{N}$. Given $n \in \mathcal{N}$, let $A = \{t \in [0,1] \mid u((1-t)y) \geq U(\mu_n)\}$ and $B = \{t \in [0,1] \mid u((1-t)y) \leq U(\mu_n)\}$. Clearly, $0 \in A$ and $1 \in B$; both A and B are closed in $[0,1]$; and $A \cup B = [0,1]$. As $[0,1]$ is connected, $A \cap B \neq \emptyset$. Let $t_n \in A \cap B$ and set $x_n = (1-t_n)y$. Clearly, $x_n \in F(\mu_n)$. If (t_n) converges to 1, then (x_n) converges to x, as required. Suppose (t_n) does not converge to 1. Then, there exists $r \in [0,1)$ and a subsequence (t_m) of (t_n) such that $(t_m) \subset [0,r]$. Therefore, $U(\mu_m) = u(x_m) = u((1-t_m)y) \geq u((1-r)y) > 0 = U(\mu)$, which contradicts the fact that $\lim_{n\uparrow\infty} \mu_n = \mu$ implies $\lim_{m\uparrow\infty} \mu_m = \mu$, and therefore, $\lim_{m\uparrow\infty} U(\mu_m) = U(\mu)$.

(2) Suppose $U(\mu) = U(\nu)$. Then, $u(x) = u(y)$, $x > 0$ and $U(\mu_n) \leq U(\mu)$ for every $n \in \mathcal{N}$. Let $A = \{t \in [0,1] \mid u(tx) \geq U(\mu_n)\}$ and $B = \{t \in [0,1] \mid u(tx) \leq U(\mu_n)\}$. Clearly, $1 \in A$ and $0 \in B$; both A and B are closed in $[0,1]$; and $A \cup B = [0,1]$. As $[0,1]$ is connected, $A \cap B \neq \emptyset$. Let $t_n \in A \cap B$ and set $x_n = t_n x$. Clearly, $x_n \in F(\mu_n)$. If (t_n) converges to 1, then (x_n) converges to x, as required. Suppose (t_n) does not converge to 1. Then, there exists $r \in [0,1)$ and a subsequence (t_m) of (t_n) such that $(t_m) \subset [0,r]$. Therefore, $U(\mu_m) = u(x_m) = u(t_m x) \leq u(rx) < u(x) = U(\mu)$, which contradicts the fact that $\lim_{n\uparrow\infty} \mu_n = \mu$ implies $\lim_{m\uparrow\infty} \mu_m = \mu$, and therefore, $\lim_{m\uparrow\infty} U(\mu_m) = U(\mu)$.

(3) Finally, suppose $U(\mu) \in (0, U(\nu))$. Then, $u(x) \in (0, u(y))$. Consider $n \in \mathcal{N}$ such that $U(\mu_n) \geq U(\mu)$. Let $A = \{t \in [0,1] \mid u(tx + (1-t)y) \geq U(\mu_n)\}$ and $B = \{t \in [0,1] \mid u(tx + (1-t)y) \leq U(\mu_n)\}$. Clearly, $0 \in A$ and $1 \in B$; both A and B are closed in $[0,1]$; and $A \cup B = [0,1]$. As $[0,1]$ is connected, $A \cap B \neq \emptyset$. Let $t_n \in A \cap B$ and set $x_n = t_n x + (1-t_n)y$. Clearly, $x_n \in F(\mu_n)$. Now consider $n \in \mathcal{N}$ such that $U(\mu_n) \leq U(\mu)$. Let $A = \{t \in [0,1] \mid u(tx) \geq U(\mu_n)\}$ and $B = \{t \in [0,1] \mid u(tx) \leq U(\mu_n)\}$. Clearly, $1 \in A$ and $0 \in B$; both A and B are closed in $[0,1]$; and $A \cup B = [0,1]$. As $[0,1]$ is connected, $A \cap B \neq \emptyset$. Let $t_n \in A \cap B$ and set $x_n = t_n x$. Clearly, $x_n \in F(\mu_n)$.

If (t_n) converges to 1, then (x_n) converges to x, as required. Suppose (t_n) does not converge to 1. Then, there exists $r \in [0,1)$ and a subsequence (t_m) of (t_n) such that $(t_m) \subset [0,r]$. Therefore, either $U(\mu_m) = u(x_m) = u(t_m x + (1-t_m)y) \geq u(rx + (1-r)y) > u(x) = U(\mu)$ or $U(\mu_m) = u(x_m) = u(t_m x) \leq u(rx) < u(x) = U(\mu)$, which contradicts the fact that $\lim_{n\uparrow\infty} \mu_n = \mu$ implies $\lim_{m\uparrow\infty} \mu_m = \mu$, and therefore, $\lim_{m\uparrow\infty} U(\mu_m) = U(\mu)$.

(B) As projections are continuous, the mapping $\pi : O \times \Delta(O) \to \Delta(O) \times O$, given by $\pi(x,\mu) = (\mu, x)$, is continuous. Then, $\operatorname{Gr} A = \{(x,\mu) \in O \times \Delta(O) \mid \mu \in A(x)\} = \{(x,\mu) \in O \times \Delta(O) \mid G \circ \pi(x,\mu) \geq 0\} = \pi^{-1} \circ G^{-1}(\Re_+)$ and $\operatorname{Gr} A^- = \{(\mu,x) \in \Delta(O) \times O \mid x \in A^-(\mu)\} = \{(\mu,x) \in \Delta(O) \times O \mid G(\mu,x) \geq 0\} = G^{-1}(\Re_+)$ are closed in $O \times \Delta(O)$ and $\Delta(O) \times O$ respectively. Therefore, as O and $\Delta(O)$ are compact, A and A^- are upper hemicontinuous. It remains to show that A and A^- are lower hemicontinuous.

To show the lower hemicontinuity of A at $x \in O$, consider a sequence $(x_n) \subset O$ converging to $x \in O$ and let $\mu \in A(x)$. By definition, $U(\mu) \geq u(x)$. We need to construct a sequence $(\mu_n) \subset \Delta(O)$ converging to μ in the weak* topology such that $\mu_n \in A(x_n)$ for every $n \in \mathcal{N}$. As U is continuous and $\Delta(O)$ is compact, there exists $\nu \in \Delta(O)$ such that $U(\nu) \geq U(\mu)$ for every $\mu \in \Delta(O)$.

Suppose $U(\mu) = U(\nu)$. Then, clearly, $u(x_n) \leq U(\mu)$ for every $n \in \mathcal{N}$. Set $\mu_n = \mu$ for every $n \in \mathcal{N}$. Clearly, $\mu_n \in A(x_n)$ for every $n \in \mathcal{N}$ and (μ_n) converges to μ.

Now suppose $U(\mu) < U(\nu)$. If $n \in \mathcal{N}$ is such that $u(x_n) \leq U(\mu)$, then set $\mu_n = t_n \mu$ where $t_n = 1$. Clearly, $\mu_n \in A(x_n)$.

Now consider $n \in \mathcal{N}$ such that $u(x_n) > U(\mu)$. Then, $U(\nu) \geq U(\delta_x) = u(x)$ for every $x \in O$. In particular, $U(\nu) \geq u(x_n) > U(\mu)$. Let $t_n \in [0,1]$ be such that $U(t_n\mu + (1-t_n)\nu) = t_n U(\mu) + (1-t_n)U(\nu) = u(x_n)$. By construction, $\mu_n = t_n\mu + (1-t_n)\nu \in A(x_n)$.

It suffices to show that (t_n) goes to 1. Note that $|U(\mu_n) - U(\mu)| \leq |u(x_n) - u(x)|$. Suppose (t_n) does not go to 1. Then, there exists $r \in [0,1)$ and a subsequence of (t_n) in $[0,r]$. As this subsequence must have a convergent subsequence, there exists a subsequence (t_m) of (t_n) such that $(t_m) \subset [0,r]$ and converges to $t \in [0,r]$. As U is continuous, $|U(t\mu + (1-t)\nu) - U(\mu)| = |U(\lim_{m\uparrow\infty}\mu_m) - U(\mu)| = |\lim_{m\uparrow\infty} U(\mu_m) - U(\mu)| = \lim_{m\uparrow\infty} |U(\mu_m) - U(\mu)|$. As $t \leq r < 1$ and $U(\nu) > U(\mu)$, we have $0 < |rU(\mu) + (1-r)U(\nu) - U(\mu)| = |U(r\mu + (1-r)\nu) - U(\mu)|$, and therefore, $0 < |U(r\mu+(1-r)\nu) - U(\mu)| \leq |U(t\mu+(1-t)\nu) - U(\mu)| = \lim_{m\uparrow\infty} |U(\mu_m) - U(\mu)| \leq \lim_{m\uparrow\infty} |u(x_m) - u(x)| = 0$, a contradiction.

To show the lower hemicontinuity of A^-, we shall use the fact that $\emptyset \neq F(\mu) \subset A^-(\mu)$ for every $\mu \in \Delta(O)$. Consider a sequence $(\mu_n) \subset \Delta(O)$ converging to $\mu \in \Delta(O)$ and let $x \in A^-(\mu)$. If $x \in F(\mu)$, then by the lower hemicontinuity of F established in (A), there exists a sequence $(x_n) \subset O$ converging to x such that $x_n \in F(\mu_n) \subset A^-(\mu_n)$ for every $n \in \mathcal{N}$. Suppose $x \in A^-(\mu) - F(\mu)$. It follows that $u(x) < U(\mu)$. As U is continuous and (μ_n) converges to μ, we have $\lim_{n\uparrow\infty} U(\mu_n) = U(\mu)$. Thus, there exists $N \in \mathcal{N}$ such that $n > N$ implies $U(\mu_n) > u(x)$, i.e., $x \in A^-(\mu_n)$. Define the sequence (x_n) as follows: for $n \leq N$, pick any $x_n \in A^-(\mu_n)$ (this selection is feasible by property (A)), and for $n > N$, set $x_n = x$. Clearly, $\lim_{n\uparrow\infty} x_n = x$ and $x_n \in A^-(\mu_n)$ for every $n \in \mathcal{N}$. ∎

Proof of Theorem 6.2. Consider $F \in \mathcal{F}$. By (A), F has nonempty values. By Theorem 6.1, F is continuous. By Theorem 3.6, there exists $u_F \in \mathcal{U}$ such that $F = \phi(u_F)$. (a) implies that $F(\mu) = \phi(u_F)(\mu)$ is closed in O. As O is compact, this means F has compact values. The result follows from the Maximum theorem (Berge [2], Section VI.3). ∎

Proof of Theorem 6.3. Consider $A \in \mathcal{A}$. By Theorem 5.5, $A = \xi(u_A)$ for some $u_A \in \mathcal{U}$. As $\delta_x \in \xi(u_A)(x)$ for every $x \in O$, we have $A(x) \neq \emptyset$ for every $x \in O$. By Theorem 6.1, A is continuous. (a) implies that u_A, and therefore U_A, is continuous. It follows that $A(x) = \xi(u_A)(x)$ is closed in $\Delta(O)$. As $\Delta(O)$ is compact, this means A has compact values. As p is continuous, the mapping $\mu \mapsto \int_O \mu(dz)\, p(z)$ is continuous. The result follows from Maximum theorem (Berge [2], Section VI.3). ∎

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