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General dualities between best replies and undominated actions

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Abstract

The central results of this paper are dualities between actions in a decision problem that are not strongly (resp., weakly) dominated over a state space and actions that are best (resp., internal-best) replies to a state. These results hold for action and state spaces that are subsets of abstract topological vector spaces, which significantly expands their set of applications in comparison to their predecessors. This is demonstrated in the game theoretic setting by applying the dualities to a player's decision problem in an abstract many-player game as well as in the σ-additive, the absolutely continuous, and the finitely additive mixed extensions of many-player games; the third extension is applicable to discontinuous games. In all these applications, the non-cooperative nature of the games is preserved by disallowing correlated decision-making by the players. The results also allow welfare theoretic applications such as the characterisation of various notions of efficient outcomes in terms of the best reply properties of the outcomes.

JEL classification: C72, D81

Key words: duality, best reply, internal-best reply, strong dominance, weak dominance, Pareto efficiency, Utilitarian efficiency

1 Introduction

This paper concerns decision problems of the form (X, Y, u) wherein X is a given decision-maker's action space, Y is the state space, and $u : X \times Y \to \Re$ yields $u(x, y)$ as the decision-maker's utility from action x in state y. The states in Y can be interpreted as the choices of other decision-makers, who may be Nature or the given decision-maker's opponents in a game.

In Section 2, we shall define the notion of an action being strongly (resp., weakly) dominated over the state space by another action and the notion

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of an action being a best (resp., an internal-best) reply to a state.[1] Our central results are Theorems 2.2 and 2.3, which are henceforth referred to as Duality I and Duality II respectively. Duality I provides conditions ensuring that an action is not strongly dominated if and only if it is a best reply. Duality II states conditions implying that an action is not weakly dominated if and only if it is an internal-best reply. Since these results admit X and Y that are subsets of abstract topological vector spaces, they significantly generalise earlier versions of these dualities and thereby greatly expand the range of their applicability. The comparisons of Dualities I and II with the earlier literature may be summarised as follows:

1. All the earlier results in the literature concern a player's decision problem in the mixed extension of a non-cooperative game. In contrast, the abstract specification of (X, Y, u) in Dualities I and II renders these results applicable to game theoretic as well as non-game theoretic decision problems, and in non-probabilistic as well as probabilistic settings. For instance, while Section 6 applies the results to non-probabilistic welfare theoretic decision problems and Section 5.1 applies them to abstract games that do not feature mixed strategies, Sections 5.2-5.5 apply them to various different models of a game's mixed extension.

2. The applicability of all the predecessors of Dualities I and II in the literature is restricted to a player's decision problem in a two-player game. When more than two players are seemingly admitted, the postulated correlation of strategy choices by a player's opponents, in effect, reduces the set of opponents to a single opponent. In contrast, as the results in Section 5 demonstrate, Dualities I and II are applicable to decision problems arising in games that have any finite number of players without having to resort to correlation of strategy choices.

3. Given a many-player game with strategy spaces that are subsets of abstract topological vector spaces, the application of Duality I to a player's decision problem in this game yields Theorem 5.1 as Duality I's analogue in this setting. This result is new and equivalent to Duality I.

   Turning to games that allow independent randomisations by the players, it is clear that, while there is essentially only one notion of a finite game's mixed extension, there can be many significantly different notions of an infinite game's mixed extension and the selection of any one is a modelling choice. While the broadest conception of a mixed extension allows a player to choose any (finitely) additive randomisation over her pure strategies, the standard probabilistic model limits

[1] An internal-best reply generalises the notion of a cautious response in Pearce [10], which is defined for decision problems (X, Y, u) wherein X and Y are the sets of probability measures over some underlying finite sets. A cautious response is a best reply in X to a measure in Y with full support.

her choice to σ-additive randomisations, or the choice may be further restricted to σ-additive randomisations that are absolutely continuous with respect to some exogenously given probability measure. The abstract nature of Duality I and Theorem 5.1 enable their application to all three models of mixed extensions.

Theorem 5.1 is first applied to a player's decision problem in the σ-additive mixed extension of a many-player game with compact metric spaces of pure strategies and continuous von Neumann-Morgenstern utilities, thereby yielding Theorem 5.6 as Duality I's analogue in this setting. This result generalises its predecessors in van Damme [5], Gale and Sherman [8], and Pearce [10], which allow only two players and finite sets of pure strategies. Since the duality claim in Zimper [13] assumes compact metric spaces of pure strategies in a two-player game, it matches the restriction of Theorem 5.6 to two-player games and is, *ipso facto*, a specialisation of Duality I and Theorem 5.1, which apply to many-player games in a much more general setting.[2]

Theorem 5.1 is also applied to a player's decision problem in the absolutely continuous mixed extension of a many-player game, which yields Theorem 5.10 as Duality I's analogue in this setting. In this mixed extension, each player chooses a probability density over pure strategies with respect to some exogenously given probability measure on the set of pure strategies, which amounts to choosing a σ-additive probability measure over pure strategies that is absolutely continuous with respect to the exogenously given measure. The duality in Theorem 5.10 does not seem to have a predecessor in the literature.

The third application of Theorem 5.1 is to a player's decision problem in the additive mixed extension of a many-player game, which yields Theorem 5.14 as Duality I's analogue in this setting. In this mixed extension, players choose additive randomisations over their pure strategies. The duality in Theorem 5.14 seems to be novel and remarkably general. For instance, it is applicable to discontinuous games.

Note that, although the sets of mixed strategies in the three models of mixed extensions may be nested for each player, the versions of Duality I flowing from these models are logically independent.

4. Moving on to Duality II, the literature does not seem to have an earlier result with anything like its scope and generality as the predecessors only address decision problems arising from the mixed extension of finite two-player games.

[2]While the claim in Zimper [13] is valid, its proof has significant gaps. Most importantly, the geometric version of the Hahn-Banach theorem that is cited in the proof does not support the claims made on its basis, such as the existence of a continuous (and implicitly, positive) linear functional that strictly separates convex sets.

Given a many-player game with strategy spaces that are subsets of abstract topological vector spaces, the application of Duality II to a player's decision problem in this game yields Theorem 5.2 as Duality II's analogue in this setting. This result is new and equivalent to Duality II.

Duality II is applicable to a player's decision problem in the σ-additive mixed extension of a many-player game with compact metric spaces of pure strategies and continuous von Neumann-Morgenstern utilities, thereby yielding Theorems 5.18 and 5.21 as Duality II's analogues in this setting. The former result states the duality between a mixed strategy's weak dominance property and its internal-best reply property when it is in the interior of the player's space of mixed strategies. The latter result derives the analogous duality for frontier points of the player's space of mixed strategies that meet a relative interiority condition. We also apply Duality II to finite many-player games and derive Theorem 5.22. This result generalises the earlier known result from finite two-player games to many-player games. It also sharpens and unifies Theorems 5.18 and 5.21 in this particular context.

Note that Duality II's technical demands make it inapplicable to a game's additive mixed extension and its applicability to a game's absolutely continuous mixed extension is a matter of investigation.

5. As noted above, earlier versions of Dualities I and II are not applicable to the mixed extensions of many-player games. Since Theorems 5.1 and 5.2 do apply to abstract many-player games, an examination of their hypotheses suggests the reason for the earlier results' limitation: in the mixed extension of a many-player game, a player's expected utility generally is not an affine function of the opponents' mixed strategy profile, and therefore it cannot generally satisfy the hypotheses of Theorems 5.1 and 5.2. The solution of this problem is evident: specify a von Neumann-Morgenstern utility such that the resulting expected utility is an affine function of the opponents' mixed strategy profile and therefore satisfies the hypotheses of Theorems 5.1 and 5.2. This tack is employed in Theorems 5.6, 5.10, 5.14, 5.18, 5.21, and 5.22. The specified von Neumann-Morgenstern utility involves no loss of generality when specialised to a two-player game.

6. Finally, we note the welfare theoretic applications in Section 6 that flow from our results, but did not do so from the earlier results. First, various notions of dominance are used to define weak and strong forms of Pareto and Utilitarian efficient outcomes. Then, Dualities I and II imply that an outcome meets one of these efficiency criteria if and only if it is optimal with respect to an appropriate welfare criterion. These characterisations (Theorems 6.2, 6.3, 6.5, and 6.6) enable the

easy identification of all efficient outcomes in multi-agent allocation models, including those with infinite-dimensional outcome spaces.

The rest of this paper is organised as follows. We state Dualities I and II in Section 2, gather many of the technical prerequisites for our work in Section 3, and prove the dualities in Section 4. Sections 5 and 6 feature the game theoretic and welfare theoretic applications respectively. Finally, we conclude the paper in Section 7 with a résumé of our findings. The proofs of lemmas, other than those in Section 4, are collected in Appendix A.

2 Dualities I and II

Our results concern the following notions.

Definition 2.1 *Consider an action space X, a state space Y, and a utility function $u : X \times Y \to \Re$. Given (X, Y, u), an action $x_0 \in X$ is said to be*

(a) strongly dominated if $u(x, .) > u(x_0, .)$ on Y for some $x \in X$,

(b) weakly dominated if $u(x, .) \geq u(x_0, .)$ and $u(x, .) \neq u(x_0, .)$ on Y for some $x \in X$,

(c) a best reply if $u(x_0, y_0) \geq u(., y_0)$ on X for some $y_0 \in Y$, and

(d) an internal-best reply if Y is a subset of a vector space and $u(x_0, y_0) \geq u(., y_0)$ on X for some y_0 that is an internal point of Y; internal points are defined and discussed in Section 3.

Duality I relates the strong dominance and best reply properties.

Theorem 2.2 *If (X, Y, u) is such that*

(a) X is a convex subset of a topological vector space,

(b) Y is a nonempty, convex, and compact subset of a locally convex topological vector space,

(c) $u : X \times Y \to \Re$ is continuous with respect to the product topology on $X \times Y$ and the Euclidean topology on $\Re$,

(d) $u(x, ty + (1 - t)y') = tu(x, y) + (1 - t)u(x, y')$ for all $x \in X$, $y, y' \in Y$, and $t \in (0, 1)$, and

(e) $u(., y)$ is quasi-concave for every $y \in Y$,

then $x_0 \in X$ is strongly dominated if and only if it is not a best reply.

Duality II relates the weak dominance and internal-best reply properties.

Theorem 2.3 *If* (X, Y, u) *and* $x_0 \in X$ *are such that*

(a) *X is a convex, compact, and metrisable subset of a Hausdorff locally convex topological vector space,*

(b) *Y is a convex, compact, and metrisable subset of a locally convex topological vector space, and Y contains an internal point,*

(c) *$u : X \times Y \to \Re$ is continuous with respect to the product topology on $X \times Y$ and the Euclidean topology on $\Re$,*

(d) *$u(x, ty + (1-t)y') = tu(x, y) + (1-t)u(x, y')$ for all $x \in X$, $y, y' \in Y$, and $t \in (0, 1)$,*

(e) *$u(tx + (1-t)x', y) = tu(x, y) + (1-t)u(x', y)$ for all $x, x' \in X$, $y \in Y$, and $t \in (0, 1)$, and*

(f) *$u(x, .) \neq u(x_0, .)$ for every $x \in X \setminus \{x_0\}$,*

then the following statements are true:

(A) *If x_0 is an interior point of X, then x_0 is weakly dominated if and only if it is not an internal-best reply.*

(B) *If x_0 is a frontier point of X and there is a compact set $C \subset X \setminus \{x_0\}$ such that, for every $x \in X \setminus \{x_0\}$, $x_0 + t(x - x_0) \in C$ for some $t \geq 1$, then x_0 is weakly dominated if and only if it is not an internal-best reply.*

We defer the proofs of the above results until Section 4. Meanwhile, we gather some technical preliminaries in the next section.

3 Notation, conventions, and facts

While many notational conventions and technical facts will be stated as and when required, the following conventions will be in force throughout this paper. $\mathcal{N}$ denotes the set of natural numbers. Given sets A and B, $A \setminus B$ is their set difference. If A and B are nonempty subsets of a vector space, then $A - B$ is their algebraic difference. Given a mapping $F : X \to 2^Y$, $F(Z) = \cup_{z \in Z} F(z)$ for $Z \subset X$ and $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ for $y \in Y$.

Given a topological space, the interior of a subset Z is $\operatorname{Int} Z$ and its frontier is $\operatorname{Fr} Z$. Unless notified otherwise, subsets of topological spaces will have the subspace topology, products of topological spaces will have the product topology, the real line $\Re$ will have the Euclidean topology, and a topological space T will have the Borel σ-algebra denoted by $\mathcal{B}(T)$. We shall abbreviate the notions of topological vector space and locally convex topological vector space to TVS and LCS respectively.

Next, we describe facts that will be used in specific sections. We start with some facts that will be used in Section 4. The first one is a coincidence result (Browder [4], Theorem 7).

Lemma 3.1 *If*

(a) X *is a nonempty and convex subset of a TVS,*

(b) Y *is a nonempty, compact, and convex subset of an LCS,*

(c) $T : X \to 2^Y$ *has a closed graph in* $X \times Y$ *and* $T(x)$ *is nonempty and convex for every* $x \in X$*, and*

(d) $S : X \to 2^Y$ *has* $S(x)$ *open in* Y *for every* $x \in X$ *and* $S^{-1}(y)$ *is nonempty and convex for every* $y \in Y$*,*

then there exists $x \in X$ *such that* $T(x) \cap S(x) \neq \emptyset$*.*[3]

Given a vector space $\mathcal{K}$ with origin $0_{\mathcal{K}}$ and $K \subset \mathcal{K}$, $x \in \mathcal{K}$ is called an internal point of K if, for every $z \in \mathcal{K}$, there exists $\epsilon > 0$ such that $|\delta| < \epsilon$ implies $x + \delta z \in K$. Let K^* be the set of internal points of K, which is called the algebraic interior of K.

If K is convex and $0_{\mathcal{K}} \in K^*$, then the support function of K is $f : \mathcal{K} \to \Re$, defined by $f(x) = \inf\{a > 0 \mid x/a \in K\}$ for $x \in \mathcal{K}$. It can be shown that $f(x) \in \Re_+$; $f(\alpha x) = \alpha f(x)$ for $\alpha \geq 0$; $f(x) \leq 1$ for $x \in K$; $f(x + y) \leq f(x) + f(y)$ for $x, y \in \mathcal{K}$; and $f(x) < 1$ if and only if $x \in K^*$ (Dunford and Schwartz [7], Lemma V.1.8).

If $\mathcal{K}$ is a TVS, then $\operatorname{Int} K \subset K^*$, and if K is convex and $\operatorname{Int} K \neq \emptyset$, then $\operatorname{Int} K = K^*$ (Dunford and Schwartz [7], Theorem V.2.1).

Lemma 3.2 *If* K *is a convex and metrisable subset of a TVS* $\mathcal{K}$ *and* $K^* \neq \emptyset$*, then* K^* *is dense in* K*.*

The notion of a set's internal approximation, as defined below, is essential for defining the concept of serial domination as *per* Definition 4.2.

Definition 3.3 *Let* $\mathcal{Y}$ *be a TVS.* $Y \subset \mathcal{Y}$ *is said to be internally approximated by a family of sets* $\{K_n \mid n \in \mathcal{N}\}$ *if*

(a) every K_n *is nonempty, convex, and compact,*

(b) $K_n \subset K_{n+1}$ *for every* $n \in \mathcal{N}$*, and*

(c) $Y^* = \cup_{n \in \mathcal{N}} K_n$*.*

[3] Since Y is compact and $\operatorname{Gr} T$ is closed in $X \times Y$, T is upper hemicontinuous, as required in Theorem 7 of Browder [4].

The following are sufficient conditions for an internally approximable set.

Lemma 3.4 *If Y is a convex and compact subset of a TVS $\mathcal{Y}$ and $Y^* \neq \emptyset$, then there is a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y.*

We now gather some facts for use in Sections 5.2 and 5.5. Consider a topological space T. Let $\mathrm{rca}(T)$ be the set of regular σ-additive measures $x : \mathcal{B}(T) \to \Re$. Given the variation norm $\|.\|_{\mathrm{var}}$, $\mathrm{rca}(T)$ is a Banach space (Dunford and Schwartz [7], p. 161). Let $\mathcal{C}(T)$ be the set of continuous and bounded functions $f : T \to \Re$. Given the supremum norm $\|.\|_{\mathrm{sup}}$, $\mathcal{C}(T)$ is a Banach space (Dunford and Schwartz [7], p. 261). If T is a compact Hausdorff space, then every continuous real-valued linear functional on $\mathcal{C}(T)$ is of the form $\mathcal{C}(T) \ni f \mapsto \int_T x(dt)\, f(t)$ for some $x \in \mathrm{rca}(T)$, and conversely, this mapping is a continuous real-valued linear functional for every $x \in \mathrm{rca}(T)$ (Dunford and Schwartz [7], Theorem IV.6.3).

Suppose T is a compact metric space and $\mathrm{rca}(T)$ is given the weak* topology. Then, $\mathrm{rca}(T)$ is a Hausdorff LCS (Dunford and Schwartz [7], Lemma V.3.3) and $(\mathcal{C}(T), \|.\|_{\mathrm{sup}})$ is separable (Aliprantis and Border [1], Lemma 3.99). Consequently, the closed unit sphere of $\mathrm{rca}(T)$ is compact and metrisable (Dunford and Schwartz [7], Theorems V.4.2 and V.5.1). Let $\Delta(T)$ be the set of σ-additive probability measures on $\mathcal{B}(T)$. Then, $\Delta(T)$ is convex, $\Delta(T) \subset \mathrm{rca}(T)$ (Parthasarathy [9], Theorem II.1.2), and $\|x\|_{\mathrm{var}} = 1$ for every $x \in \Delta(T)$. Moreover, $\Delta(T)$ is compact and metrisable (Parthasarathy [9], Theorem II.6.4), and therefore it is separable.

Finally, we note some facts employed in Section 5.4. $\mathcal{T} \subset 2^T$ is called an algebra on a set T if (a) $\emptyset \in \mathcal{T}$, (b) $T \setminus E \in \mathcal{T}$ if $E \in \mathcal{T}$, and (c) $E \cup F \in \mathcal{T}$ if $E, F \in \mathcal{T}$. If $\mathcal{T}$ is an algebra on T and $E, F \in \mathcal{T}$, then $T \in \mathcal{T}$, $E \cap F \in \mathcal{T}$, and $E \setminus F \in \mathcal{T}$.

$\mathcal{E} \subset 2^T$ is called a semialgebra on T if (a) $T, \emptyset \in \mathcal{E}$, (b) $E \cap F \in \mathcal{E}$ if $E, F \in \mathcal{E}$, and (c) if $E, F \in \mathcal{E}$ and $E \subset F$, then there are $E_0, \ldots, E_n \in \mathcal{E}$ such that $E = E_0 \subset E_1 \subset \ldots \subset E_n = F$ and $E_i \setminus E_{i-1} \in \mathcal{E}$ for $i = 1, \ldots, n$.

Lemma 3.5 *Consider a semialgebra $\mathcal{E}$ on a set T. If $\mathcal{A}$ is the collection of finite unions of pairwise disjoint sets from $\mathcal{E}$, then $\mathcal{A}$ is an algebra on T (called the algebra generated by $\mathcal{E}$) that equals the intersection of all algebras on T that have $\mathcal{E}$ as a sub-collection.*

Consider an algebra $\mathcal{T}$ on a set T. Let $\mathrm{ba}(T, \mathcal{T})$ be the set of bounded additive measures $x : \mathcal{T} \to \Re$. Given the variation norm $\|.\|_{\mathrm{var}}$, $\mathrm{ba}(T, \mathcal{T})$ is a Banach space (Dunford and Schwartz [7], p. 161). Let $B(T, \mathcal{T})$ be the set of uniform limits of step functions $\sum_{j=1}^{n} a_j 1_{E_j}$, where $a_1, \ldots, a_n \in \Re$, $E_1, \ldots, E_n \in \mathcal{T}$, and 1_{E_j} is the indicator function of E_j. Given the supremum norm $\|.\|_{\mathrm{sup}}$, $B(T, \mathcal{T})$ is a Banach space (Dunford and Schwartz [7],

p. 258). Every continuous real-valued linear functional on $B(T,\mathcal{T})$ is of the form $B(T,\mathcal{T}) \ni f \mapsto \int_T x(dt)\, f(t)$ for some $x \in \mathrm{ba}(T,\mathcal{T})$, and conversely, this mapping is a continuous real-valued linear functional for every $x \in \mathrm{ba}(T,\mathcal{T})$ (Dunford and Schwartz [7], Theorem IV.5.1); see Section III.2 in Dunford and Schwartz [7] for the notion of integration with respect to additive measures.

Suppose $\mathrm{ba}(T,\mathcal{T})$ is given the weak* topology. Then, $\mathrm{ba}(T,\mathcal{T})$ is a Hausdorff LCS (Dunford and Schwartz [7], Lemma V.3.3) and its closed unit sphere is compact (Dunford and Schwartz, Theorem V.4.2). Let $\Delta(\mathcal{T}) = \{x \in \mathrm{ba}(T,\mathcal{T}) \mid x \geq 0 \ \wedge \ x(T) = 1\}$. Then, $\Delta(\mathcal{T})$ is convex and closed.[4] Since $\Delta(\mathcal{T})$ is contained in the closed unit sphere of $\mathrm{ba}(T,\mathcal{T})$, it is compact.

4 Proofs of Dualities I and II

We start with Duality I relating strong dominance and best replies. In addition to being important in its own right, this result will play a crucial role in the proof of Theorem 2.3 *via* its application in Lemma 4.3.

Proof of Theorem 2.2 Consider (X,Y,u) *ex hypothesi.* It follows from the definitions that, if x_0 is strongly dominated, then x_0 cannot be a best reply.

Conversely, suppose x_0 is not strongly dominated and not a best reply. Define $T : X \to 2^Y$ and $S : X \to 2^Y$ by $T(x) = \{y \in Y \mid u(x_0,y) \geq u(x,y)\}$ and $S(x) = \{y \in Y \mid u(x_0,y) < u(x,y)\}$ respectively.

Since $x_0 \in X$, we have $X \neq \emptyset$. Therefore, hypotheses (a) and (b) imply that hypotheses (a) and (b) of Lemma 3.1 are satisfied.

Hypothesis (c) implies that $\mathrm{Gr}\, T = \{(x,y) \in X \times Y \mid u(x_0,y) \geq u(x,y)\}$ is closed in $X \times Y$. As x_0 is not strongly dominated, $T(x) \neq \emptyset$ for every $x \in X$. For every $x \in X$, hypothesis (d) implies that $T(x)$ is convex. So, hypothesis (c) of Lemma 3.1 is satisfied.

Hypothesis (c) implies that $S(x)$ is open in Y for every $x \in X$. As x_0 is not a best reply, $S^{-1}(y) = \{x \in X \mid u(x_0,y) < u(x,y)\} \neq \emptyset$ for every $y \in Y$. For every $y \in Y$, hypothesis (e) implies that $S^{-1}(y)$ is convex. So, hypothesis (d) of Lemma 3.1 is satisfied.

By Lemma 3.1, $T(x) \cap S(x) \neq \emptyset$ for some $x \in X$, a contradiction. ■

Next, we turn to Theorem 2.3, which relates internal-best replies and weak dominance. These notions are bridged *via* the notion of a decision-maker's action being serially dominated. The plan for doing this is as follows:

1. Lemma 4.1 shows that, given $Y^* \neq \emptyset$ and the hypotheses of Theorem 2.2, an action is weakly dominated on Y if and only if it is strongly dominated on Y^*.

[4] Let $x \in \mathrm{ba}(T,\mathcal{T})$ be an accumulation point of $\Delta(\mathcal{T})$. Then, there is a net $(x^\alpha) \subset \Delta(\mathcal{T})$ converging to x. As $1_E \in B(T,\mathcal{T})$ for $E \in \mathcal{T}$, $\lim_\alpha x^\alpha(E) = \lim_\alpha \int_T x^\alpha(dt)\, 1_E(t) = \int_T x(dt)\, 1_E(t) = x(E)$. It follows that $x \geq 0$ and $x(T) = 1$. Hence, $x \in \Delta(\mathcal{T})$.

2. Using the internal approximation of Y by a family of sets $\{K_n \mid n \in \mathcal{N}\}$, Definition 4.2 formalises the notion of action x_0 being serially dominated by a sequence of actions (x_n), where each x_n strongly dominates x_0 on K_n.

3. Lemma 4.3 shows that, given the hypotheses of Theorem 2.2, x_0 is serially dominated if and only if it is not an internal-best reply.

4. Lemma 4.4 provides sufficient conditions for an action to be weakly dominated if and only if it is serially dominated.

5. The proof of Theorem 2.3 applies Lemmas 4.3 and 4.4.

We proceed to the first step of the plan outlined above.

Lemma 4.1 *Suppose (X, Y, u) satisfies the hypotheses of Theorem 2.2. If Y is metrisable and $Y^* \neq \emptyset$, then $x_0 \in X$ is weakly dominated if and only if there exists $x \in X$ such that $u(x_0, .) < u(x, .)$ on Y^*.*

Proof. Suppose x_0 is weakly dominated by $x \in X$. Then, there exists $y_0 \in Y$ such that $u(x_0, y_0) < u(x, y_0)$. Consider $y \in Y^*$. As $y - y_0 \in \mathcal{Y}$, there exists $\epsilon > 0$ such that $y + \delta(y - y_0) \in Y$ for every $\delta \in (0, \epsilon)$. Let $\delta_0 \in (0, \epsilon)$. Then, $y_1 = y + \delta_0(y - y_0) \in Y$ and $y = ty_1 + (1 - t)y_0 \in Y$, where $t = (1 + \delta_0)^{-1} \in (0, 1)$. As $u(x_0, y_1) \leq u(x, y_1)$, hypothesis (d) implies $u(x_0, y) = u(x_0, ty_1 + (1 - t)y_0) = tu(x_0, y_1) + (1 - t)u(x_0, y_0) < tu(x, y_1)+(1-t)u(x, y_0) = u(x, ty_1+(1-t)y_0) = u(x, y)$. So, $u(x_0, .) < u(x, .)$ on Y^*.

Conversely, suppose there exists $x \in X$ such that $u(x_0, .) < u(x, .)$ on Y^*. Consider $y \in Y$. As Y is convex, metrisable, and $Y^* \neq \emptyset$, Lemma 3.2 implies that Y^* is dense in Y. So, there is a sequence $(y_n) \subset Y^*$ converging to y. As $u(x_0, y_n) < u(x, y_n)$ for every n and u is continuous, we have $u(x_0, y) \leq u(x, y)$. As $Y^* \neq \emptyset$, there exists $y_0 \in Y^*$ and $u(x_0, y_0) < u(x, y_0)$. So, x_0 is weakly dominated. ■

We are ready for the second step of defining serial domination.

Definition 4.2 *Consider (X, Y, u). $x_0 \in X$ is said to be serially dominated if, for every family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y, and for every $n \in \mathcal{N}$, there exists $x \in X$ such that $u(x_0, .) < u(x, .)$ on K_n.*

The next step establishes the duality between serial domination and internal-best replies.

Lemma 4.3 *If (X, Y, u) satisfies the hypotheses of Theorem 2.2, then $x_0 \in X$ is not serially dominated if and only if it is an internal-best reply.*

Proof. If $Y^* = \emptyset$, then x_0 is not an internal-best reply and x_0 is serially dominated because there is no family of sets that internally approximates Y. Henceforth, let $Y^* \neq \emptyset$.

Suppose x_0 is not serially dominated for (X, Y, u). Then, there exists a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y, and there exists $n \in \mathcal{N}$ such that, for every $x \in X$, $u(x_0, y) \geq u(x, y)$ for some $y \in K_n$. Using Definition 3.3, (X, K_n, u) satisfies the hypotheses of Theorem 2.2. Therefore, x_0 is not strongly dominated with respect to (X, K_n, u). By Theorem 2.2, x_0 is a best reply with respect to (X, K_n, u), i.e., $u(x_0, y) \geq u(., y)$ on X for some $y \in K_n$. As $y \in K_n \subset Y^*$, x_0 is an internal-best reply for (X, Y, u).

Conversely, suppose x_0 is an internal-best reply and it is serially dominated with respect to (X, Y, u). As x_0 is an internal-best reply, $u(x_0, y) \geq u(., y)$ on X for some $y \in Y^*$. As $Y^* \neq \emptyset$, Lemma 3.4 and hypothesis (b) of Theorem 2.2 imply that there is a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y. As x_0 is serially dominated, for every $n \in \mathcal{N}$, there exists $x_n \in X$ such that $u(x_0, .) < u(x_n, .)$ on K_n. As $y \in Y^* = \cup_{n \in \mathcal{N}} K_n$, it follows that $y \in K_n$ for some $n \in \mathcal{N}$. Consequently, $u(x_n, y) \leq u(x_0, y) < u(x_n, y)$, a contradiction. ■

The fourth step provides conditions for the equivalence of serial domination and weak domination.

Lemma 4.4 *Consider (X, Y, u) and $x_0 \in X$ that satisfy the hypotheses of Theorem 2.3. If there is a compact set $C \subset X \setminus \{x_0\}$ such that, for every $x \in X \setminus \{x_0\}$, $h(t) = x_0 + t(x - x_0) \in C$ for some $t \geq 1$, then x_0 is weakly dominated if and only if it is serially dominated.*

Proof. Since the hypotheses of Theorem 2.3 are satisfied, so are the hypotheses of Theorem 2.2.

Suppose x_0 is weakly dominated. By Lemma 4.1, there exists $x \in X$ such that $u(x_0, .) < u(x, .)$ on Y^*. Consider a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates Y. For every $n \in \mathcal{N}$, as $K_n \subset Y^*$, we have $u(x_0, .) < u(x, .)$ on K_n. So, x_0 is serially dominated.

Conversely, suppose x_0 is not weakly dominated. We show that x_0 is not serially dominated.

1. Consider $x \in C$. Then, $x \neq x_0$. If $u(x_0, .) \leq u(x, .)$ on Y, then $u(x, .) \neq u(x_0, .)$ implies $u(x_0, y) < u(x, y)$ for some $y \in Y$. Thus, x_0 is weakly dominated, which is a contradiction. So, $u(x_0, y) > u(x, y)$ for some $y \in Y$. As Y^* is dense in Y by Lemma 3.2 and u is continuous by hypothesis (c) of Theorem 2.3, $u(x_0, y(x)) > u(x, y(x))$ for some $y(x) \in Y^*$. As u is continuous, there is an open neighbourhood $V(x)$ of x such that $u(x_0, y(x)) > u(., y(x))$ on $V(x) \cap X$.

   The collection $\mathcal{V} = \{V(x') \mid x' \in C\}$ is an open cover of C. As C is compact, $\mathcal{V}$ contains a finite subcover of C, say $\{V(x_1), \ldots, V(x_n)\}$ for some $\{x_1, \ldots, x_n\} \subset C$.

It follows that $x \in V(x_j)$ for some $j \in \{1, \dots n\}$. Since $u(x_0, y(x_j)) > u(., y(x_j))$ on $V(x_j) \cap X$, we have $u(x_0, y(x_j)) > u(x, y(x_j))$.

Thus, for every $x \in C$, there exists $y_x \in \{y(x_1), \dots, y(x_n)\} \subset Y^*$ such that $u(x_0, y_x) > u(x, y_x)$.

2. Let $x \in X \setminus \{x_0\}$. By assumption, $h(t) \in C$ for some $t \geq 1$. By step 1, there exists $y_{h(t)} \in \{y(x_1), \dots, y(x_n)\}$ such that $u(x_0, y_{h(t)}) > u(h(t), y_{h(t)})$. As $t \geq 1$, $x = t^{-1}h(t) + (1 - t^{-1})x_0$ and $u(x, y_{h(t)}) = (1 - t^{-1})u(x_0, y_{h(t)}) + t^{-1}u(h(t), y_{h(t)}) < u(x_0, y_{h(t)})$.

   Thus, for every $x \in X \setminus \{x_0\}$, there exists $y \in \{y(x_1), \dots, y(x_n)\} \subset Y^*$ such that $u(x_0, y) > u(x, y)$.

3. By hypothesis (b) of Theorem 2.3 and Lemma 3.4, Y is internally approximated by a family of sets $\{K_m \mid m \in \mathcal{N}\}$. It follows that $\{y(x_1), \dots, y(x_n)\} \subset Y^* = \cup_{m \in \mathcal{N}} K_m$ and $K_m \subset K_{m+1}$ for every $m \in \mathcal{N}$.

   Thus, $\{y(x_1), \dots, y(x_n)\} \subset K_m$ for some $m \in \mathcal{N}$.

4. Suppose x_0 is serially dominated. Then, there exists $x^* \in X$ such that $u(x_0, .) < u(x^*, .)$ on K_m. So, $x^* \in X \setminus \{x_0\}$. By steps 2 and 3, there exists $y^* \in \{y(x_1), \dots, y(x_n)\} \subset K_m$ such that $u(x^*, y^*) < u(x_0, y^*)$. As $y^* \in K_m$, we have $u(x^*, y^*) < u(x_0, y^*) < u(x^*, y^*)$, which is a contradiction. ■

Finally, we arrive at the proof of the duality between weak domination and internal-best replies.

Proof of Theorem 2.3 Consider (X, Y, u) and x_0 *ex hypothesi*. Let $\mathcal{X}$ be the TVS mentioned in hypothesis (a).

(A) As X is compact and $\mathcal{X}$ is Hausdorff, X is closed in $\mathcal{X}$. Since $x_0 \in \operatorname{Int} X$, we have $C = \operatorname{Fr} X \subset X \setminus \{x_0\}$. As C is a closed subset of the compact set X, C is compact. We show that, for every $x \in X \setminus \{x_0\}$, $x_0 + t(x - x_0) \in C$ for some $t \geq 1$. Then, the result follows from Lemmas 4.3 and 4.4.

Consider $x \in X \setminus \{x_0\}$. Define $h : [1, \infty) \to \mathcal{X}$ by $h(t) = x_0 + t(x - x_0)$ and $g : [1, \infty) \to \mathcal{X}$ by $g(t) = t(x - x_0)$. Then, $h(.) = x_0 + g(.)$, $h^{-1}(X) = g^{-1}(X - \{x_0\})$, and $1 \in h^{-1}(X)$. As $\mathcal{X}$ is a TVS, h is continuous. So, $h^{-1}(X)$ is closed in $[1, \infty)$. Suppose $g^{-1}(X - \{x_0\})$ is bounded. Then, $h^{-1}(X)$ is nonempty and compact, and there exists $t = \max h^{-1}(X)$. As $h(t) \in X$ and $h(t + \epsilon) \notin X$ for every $\epsilon > 0$, we have $h(t) \in \operatorname{Fr} X = C$.

It remains to show that $g^{-1}(X - \{x_0\})$ is bounded. Suppose it is not bounded. Then, there exists an unbounded, strictly increasing sequence $(t_n) \subset [1, \infty)$ such that $g(t_n) = t_n(x - x_0) \in X - \{x_0\}$ for every $n \in \mathcal{N}$. Given hypothesis (a), the topology of $\mathcal{X}$ is the projective topology generated by a total family of seminorms $\mathcal{P}$, i.e., the coarsest topology on the underlying vector space that makes every $p \in \mathcal{P}$ continuous. Since $\mathcal{P}$ is total and $x \neq x_0$, we have $p(x - x_0) > 0$ for some $p \in \mathcal{P}$. Then, $p \circ g(t_n) = t_n p(x - x_0) > 0$ for every $n \in \mathcal{N}$ and $\{p \circ g(t_n) \mid n \in \mathcal{N}\}$ is unbounded above. As p is continuous and (t_n) is unbounded, $\{p^{-1}([0, t_n)) \mid n \in \mathcal{N}\}$ is an open cover of $X - \{x_0\}$. Given hypothesis (a), as X is compact, $X - \{x_0\}$ is compact. Therefore, $\{p^{-1}([0, t_n)) \mid n \in \mathcal{N}\}$ contains a finite subcover. So, $\{g(t_n) \mid n \in \mathcal{N}\} \subset X - \{x_0\} \subset p^{-1}([0, t_{n_0}))$ for some $n_0 \in \mathcal{N}$. This implies $p \circ g(t_n) < t_{n_0}$ for every $n \in \mathcal{N}$, which is a contradiction.

(B) Combine Lemmas 4.3 and 4.4. ■

5 Game theoretic applications

In this section, we shall execute the programme outlined in Section 1 concerning game theoretic applications of Dualities I and II.

5.1 Abstract game and Dualities I and II

Consider a game $\Gamma = \{N, (S_j, u_j)_{j \in N}\}$ where N is the set of players, S_j is player j's strategy space, and $u_j : \prod_{k \in N} S_k \to \Re$ is player j's utility function. Given $i \in N$ and setting $S_{-i} = \prod_{j \in N \setminus \{i\}} S_j$, player i's decision problem in Γ is (S_i, S_{-i}, u_i). Applying Theorem 2.2 to $(X, Y, u) \equiv (S_i, S_{-i}, u_i)$, we have

Theorem 5.1 *If $\Gamma = \{N, (S_j, u_j)_{j \in N}\}$ and $i \in N$ are such that*

(a) N is finite,

(b) S_i is a convex subset of a TVS,

(c) for every $j \in N \setminus \{i\}$, S_j is a nonempty, convex, and compact subset of an LCS,

(d) u_i is continuous,

(e) $u_i(x, ty + (1-t)y') = tu_i(x, y) + (1-t)u_i(x, y')$ for all $x \in S_i$, $y, y' \in S_{-i}$, and $t \in (0, 1)$, and

(f) $u_i(., y)$ is quasi-concave for every $y \in S_{-i}$,

then $x_0 \in S_i$ is strongly dominated with respect to (S_i, S_{-i}, u_i) if and only if it is not a best reply.

The required properties of $Y \equiv S_{-i}$ are inherited from the properties of S_j, $j \in N \setminus \{i\}$, as *per* hypothesis (c). Similarly, applying Theorem 2.3 to $(X, Y, u) \equiv (S_i, S_{-i}, u_i)$, we have

Theorem 5.2 *If $\Gamma = \{N, (S_j, u_j)_{j \in N}\}$, $i \in N$, and $x_0 \in S_i$ are such that*

(a) N is finite,

(b) S_i is a convex, compact, and metrisable subset of a Hausdorff LCS,

(c) for every $j \in N \setminus \{i\}$, S_j is a convex, compact, and metrisable subset of an LCS, with $S_j^* \neq \emptyset$,

(d) u_i is continuous,

(e) $u_i(x, ty+(1-t)y') = tu_i(x, y)+(1-t)u_i(x, y')$ for all $x \in S_i$, $y, y' \in S_{-i}$, and $t \in (0, 1)$,

(f) $u_i(tx + (1 - t)x', y) = tu_i(x, y) + (1 - t)u_i(x', y)$ for all $x, x' \in S_i$, $y \in S_{-i}$, and $t \in (0, 1)$, and

(g) $u_i(x, .) \neq u_i(x_0, .)$ for every $x \in S_i \setminus \{x_0\}$,

then the following statements are true with respect to (S_i, S_{-i}, u_i):

(A) If $x_0 \in \operatorname{Int} S_i$, then x_0 is weakly dominated if and only if it is not an internal-best reply.

(B) If $x_0 \in \operatorname{Fr} S_i$ and there is a compact set $S_i' \subset S_i \setminus \{x_0\}$ such that, for every $x \in S_i \setminus \{x_0\}$, $x_0 + t(x - x_0) \in S_i'$ for some $t \geq 1$, then x_0 is weakly dominated if and only if it is not an internal-best reply.

Remark 5.3 *Since N is finite, it is easily verified that $S_{-i}^* \neq \emptyset$ if and only if $S_j^* \neq \emptyset$ for every $j \in N \setminus \{i\}$. Thus, hypotheses (a) and (c) imply $S_{-i}^* \neq \emptyset$.*

Remark 5.4 *Setting $|N| = 2$ in the above two results reveals that Theorem 5.1 (resp. 5.2) is equivalent to Theorem 2.2 (resp. 2.3).*

5.2 Duality I and σ-additive mixed extensions

Now consider a game $\Gamma = \{N, (C_j, v_j)_{j \in N}\}$ that may not satisfy the hypotheses of Theorems 5.1 and 5.2. So, we take recourse to Γ's σ-additive mixed extension $\Gamma_\sigma(m)$ by interpreting C_j as the set of player j's pure strategies and $v_j : C \to \Re$ as player j's von Neumann-Morgenstern utility.

Definition 5.5 *Consider* $\Gamma = \{N, (C_j, v_j)_{j\in N}\}$ *and* $i \in N$ *such that:* N *is finite;* C_j *is a nonempty, compact, metric space for every* $j \in N$*; setting* $C = \prod_{k\in N} C_k$, $v_j : C \to \Re$ *is continuous for every* $j \in N \setminus \{i\}$ *and* $v_i : C \to \Re$ *is defined by* $v_i = \sum_{j\in N\setminus\{i\}} w_j \circ \pi_{i,j}$*, where* $\pi_{i,j}$ *projects* C *on* $C_i \times C_j$ *and* $w_j : C_i \times C_j \to \Re$ *is continuous for every* $j \in N \setminus \{i\}$.

(a) $\Gamma_\sigma(m) = \{N, (\Delta(C_j), V_j)_{j\in N}\}$ *is the* σ*-additive mixed extension of* Γ *where, for every* $j \in N$,

  (i) $\Delta(C_j)$ *is the set of* σ*-additive probability measures on* $(C_j, \mathcal{B}(C_j))$,

  (ii) $V_j : \prod_{k\in N} \Delta(C_k) \to \Re$ *is given by* $V_j(y) = \int_C \bar{y}(dc)\, v_j(c)$ *for* $y = (y_k)_{k\in N} \in \prod_{k\in N} \Delta(C_k)$ *and* $\bar{y} = \prod_{k\in N} y_k$.

(b) $(\Delta(C_i), \prod_{j\in N\setminus\{i\}} \Delta(C_j), V_i)$ *is player* i*'s decision problem in* $\Gamma_\sigma(m)$.

Note that player i's opponents randomise *independently* over their pure strategies. In order to motivate the hypotheses on v_i, set $(S_i, S_{-i}, u_i) \equiv (\Delta(C_i), \prod_{j\in N\setminus\{i\}} \Delta(C_j), V_i)$. For every $j \in N \setminus \{i\}$, as $\pi_{i,j}$ and w_j are continuous, so is $w_j \circ \pi_{i,j}$. Therefore, as N is finite, v_i is continuous on C.

Suppose $|N| = 2$. Then, $\pi_{i,j}$ amounts to the identity mapping on C and the assumed v_i can be any continuous real-valued function on C. Moreover, $u_i \equiv V_i$ satisfies hypothesis (e) of Theorems 5.1 and 5.2.

Now suppose $|N| > 2$. If $v_i : C \to \Re$ is some arbitrary continuous function, then $u_i \equiv V_i$ may not satisfy hypothesis (e) of Theorems 5.1 and 5.2. The reason is that the mapping $\prod_{j\in N\setminus\{i\}} \Delta(C_j) \ni (y_j)_{j\in N\setminus\{i\}} \mapsto \prod_{j\in N\setminus\{i\}} y_j \in \Delta(\prod_{j\in N\setminus\{i\}} C_j)$ is generally not affine when $|N| > 2$. As is shown below, the assumed v_i resolves this problem.

Theorem 5.6 *If* $(\Delta(C_i), \prod_{j\in N\setminus\{i\}} \Delta(C_j), V_i)$ *is defined by Definition 5.5, then* $x_0 \in \Delta(C_i)$ *is strongly dominated if and only if it is not a best reply.*

Proof. It suffices to verify that $\{N, (\Delta(C_j), V_j)_{j\in N}\}$ satisfies Theorem 5.1's hypotheses.

1. As N is finite, hypothesis (a) of Theorem 5.1 is satisfied.

2. As *per* the discussion in Section 3, $\text{rca}(C_j)$ with its weak* topology is a Hausdorff LCS for every $j \in N$. Moreover, $\Delta(C_j)$ is a convex, compact, and metrisable subset of $\text{rca}(C_j)$ for every $j \in N$. So, hypotheses (b) and (c) of Theorem 5.1 are satisfied.

3. Since N is finite and $\Delta(C_j)$ is metrisable for every $j \in N$, the product topology of $\prod_{j\in N} \Delta(C_j)$ is metrisable.

4. Let a sequence $(y^n) \subset \prod_{j\in N} \Delta(C_j)$ converge to $y \in \prod_{j\in N} \Delta(C_j)$ in the product topology of $\prod_{j\in N} \Delta(C_j)$. Then, $(y_j^n) \subset \Delta(C_j)$ weak* converges to $y_j \in \Delta(C_j)$ for every $j \in N$. As each C_j is compact metric, so is C. Therefore, C is separable. Let $\bar{y}^n = \prod_{j\in N} y_j^n \in \Delta(C)$ for every $n \in \mathcal{N}$. Using Theorem 2.8 in Billingsley [2], the sequence $(\bar{y}^n) \subset \Delta(C)$ weak* converges to $\bar{y} = \prod_{j\in N} y_j \in \Delta(C)$. As v_i is continuous – and therefore, bounded – on C, $V_i(y^n) = \int_C \bar{y}^n(dc)\, v_i(c) \to \int_C \bar{y}(dc)\, v_i(c) = V_i(y)$. Using step 3, this implies V_i is continuous and satisfies hypothesis (d) of Theorem 5.1.

5. Consider $(x, y) \in \Delta(C_i) \times \prod_{j\in N\setminus\{i\}} \Delta(C_j)$ and let $\bar{y} = \prod_{j\in N\setminus\{i\}} y_j$. Then, $V_i(x, y) = \int_C x \times \bar{y}(dc) \sum_{j\in N\setminus\{i\}} w_j \circ \pi_{i,j}(c) = \sum_{j\in N\setminus\{i\}} \int_C x \times \bar{y}(dc)\, w_j \circ \pi_{i,j}(c)$. Changing variables, $\int_C x \times \bar{y}(dc)\, w_j \circ \pi_{i,j}(c) = \int_{C_i\times C_j} (x\times\bar{y})\circ\pi_{i,j}^{-1}(dc_i, dc_j)\, w_j(c_i, c_j) = \int_{C_i\times C_j} x\times y_j(dc_i, dc_j)\, w_j(c_i, c_j)$ for every $j \in N \setminus \{i\}$.

   So, $V_i(x, y) = \sum_{j\in N\setminus\{i\}} \int_{C_i\times C_j} x \times y_j(dc_i, dc_j)\, w_j(c_i, c_j)$. Using this formula, if $y, z \in \prod_{j\in N\setminus\{i\}} \Delta(C_j)$ and $t \in (0, 1)$, then it can be readily checked that $V_i(x, ty + (1 - t)z) = tV_i(x, y) + (1 - t)V_i(x, z)$. So, hypothesis (e) of Theorem 5.1 is satisfied.

6. Analogously, $V_i(tx + (1 - t)x', y) = tV_i(x, y) + (1 - t)V_i(x', y)$ for all $x, x' \in \Delta(C_i)$, $y \in \prod_{j\in N\setminus\{i\}} \Delta(C_j)$, and $t \in (0, 1)$. So, hypothesis (f) of Theorem 5.1 is satisfied. ■

The above result clearly applies if $\Gamma = \{N, (C_j, v_j)_{j\in N}\}$ is a finite game.

5.3 Duality I and absolutely continuous mixed extensions

We shall now apply Theorem 5.1 – or equivalently, Duality I – to a game's absolutely continuous mixed extension. This extension restricts a player's admissible randomisations over the pure strategies to those that are absolutely continuous relative to an exogenously given probability measure. Equivalently, the player is restricted to admissible densities relative to the exogenous measure; the equivalence follows from the Radon-Nikodým theorem (Rao [11], Theorem 5.3.3). Since admitting all integrable densities makes the application intractable, we systematically impose restrictions on the set of admissible densities as follows.

Let $p \in (1, \infty)$. Consider a probability space $(\Omega, \mathcal{F}, P)$. Let $L^p(\Omega, \mathcal{F}, P)$ be the set of measurable functions $x : \Omega \to \Re$ such that $\int_\Omega P(d\omega)\, |x(\omega)|^p < \infty$; we shall abbreviate $L^p(\Omega, \mathcal{F}, P)$ to L^p_Ω. Given the topology generated by the norm $\|x\|_{p,\Omega} = [\int_\Omega P(d\omega)\, |x(\omega)|^p]^{1/p}$ for $x \in L^p_\Omega$, L^p_Ω is a Banach space; we shall refer to $(\Omega, \mathcal{F}, P)$ as separable if this Banach space is separable.[5]

[5]Less reductively, one may provide conditions on $(\Omega, \mathcal{F}, P)$ that ensure the separability

Given the closed unit sphere $B_\Omega^p = \{x \in L_\Omega^p \mid \|x\|_{p,\Omega} \leq 1\}$, we shall refer to

$$D = \bigcap_{E \in \mathcal{F}} \left\{ x \in B_\Omega^p \,\middle|\, \int_\Omega P(d\omega)\, x(\omega) = 1 \ \wedge \ \int_E P(d\omega)\, x(\omega) \geq 0 \right\} \tag{1}$$

as the admissible set of densities relative to $(\Omega, \mathcal{F}, P)$. If $x \in D$, then $x \geq 0$ P-a.e. The condition $D \subset B_\Omega^p$ amounts to assuming the norm-boundedness of D; the bound 1 is arbitrary and it may be replaced with any $r > 0$ without affecting the results. A density $x \in D$ generates the σ-additive probability measure $\mathcal{F} \ni E \mapsto \int_E P(d\omega)\, x(\omega) \in \Re$, which is absolutely continuous with respect to P, i.e., $P(E) = 0$ implies $\int_E P(d\omega)\, x(\omega) = 0$. We note the key properties of L_Ω^p and D for our application.

Lemma 5.7 *Consider a separable probability space $(\Omega, \mathcal{F}, P)$. Given its weak topology, L_Ω^p is a Hausdorff LCS and the admissible set of densities D, defined by (1), is nonempty, convex, metrisable, closed, and compact.*

Now consider probability spaces $(\Omega, \mathcal{F}, P)$ and $(\Theta, \mathcal{T}, Q)$. Given this pair, $\Omega \times \Theta$ will be given the product σ-algebra $\mathcal{F} \times \mathcal{T}$ and the product measure $P \times Q$, thereby generating the probability space $(\Omega \times \Theta, \mathcal{F} \times \mathcal{T}, P \times Q)$ and the Banach space $(L_{\Omega\times\Theta}^p, \|.\|_{p,\Omega\times\Theta})$.

Lemma 5.8 *Consider separable probability spaces $(\Omega, \mathcal{F}, P)$ and $(\Theta, \mathcal{T}, Q)$ with the admissible sets of densities X and Y respectively, as per Equation (1). Suppose $w : \Omega \times \Theta \to \Re$ is $\mathcal{F} \times \mathcal{T}$ measurable and bounded. Let $\varphi(\omega, y) = \int_\Theta Q(d\theta)\, w(\omega, \theta) y(\theta)$ for $(\omega, y) \in \Omega \times Y$ and $\Phi(x, y) = \int_{\Omega\times\Theta} P \times Q(d\omega, d\theta)\, w(\omega, \theta) x(\omega) y(\theta)$ for $(x, y) \in X \times Y$. If L_Ω^p, L_Θ^p, and $L_{\Omega\times\Theta}^p$ are given their respective weak topologies, then:*

(A) φ is real-valued and bounded, $\varphi(\omega, .)$ is continuous for $\omega \in \Omega$, and $\varphi(., y)$ is measurable for $y \in Y$.

(B) Φ is real-valued, bounded, $\Phi(x, y) = \int_\Omega P(d\omega)\, x(\omega) \varphi(\omega, y)$ for $(x, y) \in X \times Y$, $\Phi(x, .)$ is continuous on Y for $x \in X$, and $\Phi(., y)$ is continuous on X for $y \in Y$.

(C) If the family of functions $\{\varphi(\omega, .) \mid \omega \in \Omega\}$ is pointwise equicontinuous, then Φ is continuous.

(D) $\Phi(x, ty + (1-t)y') = t\Phi(x, y) + (1-t)\Phi(x, y')$ and $\Phi(tx + (1-t)x', y) = t\Phi(x, y) + (1-t)\Phi(x', y)$ for all $x, x' \in X$, $y, y' \in Y$, and $t \in (0, 1)$.

Given the above formalism, the absolutely continuous mixed extension $\Gamma_{ac}(m)$ of a game Γ and player i's decision problem in $\Gamma_{ac}(m)$ are as follows.

of $(L_\Omega^p, \|.\|_{p,\Omega})$. For instance, if Ω is a separable metric space (e.g., a subspace of a Euclidean space) and $\mathcal{F} = \mathcal{B}(\Omega)$, then $(L_\Omega^p, \|.\|_{p,\Omega})$ is separable (Brezis [3], Theorem 4.13).

Definition 5.9 *Consider* $\Gamma = \{N, (C_j, v_j)_{j\in N}\}$*, separable probability spaces* $\{(C_j, \mathcal{C}_j, \mu_j) \mid j \in N\}$*, and* $i \in N$*. Suppose* N *is finite. Given the product measurable space* $(C, \mathcal{C}) \equiv (\prod_{k\in N} C_k, \prod_{k\in N} \mathcal{C}_k)$*, suppose* $v_j : C \to \Re$ *is measurable and bounded for every* $j \in N \setminus \{i\}$ *and* $v_i : C \to \Re$ *is given by* $v_i = \sum_{j\in N\setminus\{i\}} w_j \circ \pi_{i,j}$*, where* $\pi_{i,j}$ *projects* C *on* $C_i \times C_j$ *and* $w_j : C_i \times C_j \to \Re$ *is* $\mathcal{C}_i \times \mathcal{C}_j$ *measurable and bounded for every* $j \in N \setminus \{i\}$*.*

(a) $\Gamma_{ac}(m) = \{N, (S_j, u_j)_{j\in N}\}$ *is the absolutely continuous mixed extension of* Γ *relative to the probability spaces* $\{(C_j, \mathcal{C}_j, \mu_j) \mid j \in N\}$ *if, for every* $j \in N$*,*

(i) S_j *is the admissible set of densities relative to* $(C_j, \mathcal{C}_j, \mu_j)$ *as per Equation (1),*

(ii) $u_j(s) = \int_C \bar{\mu}(dc)\, v_j(c)\bar{s}(c)$ *for* $s \in S \equiv \prod_{k\in N} S_k$ *defines* $u_j : S \to \Re$*, where* $\bar{s}(c) = \prod_{k\in N} s_k(c_k)$ *and* $\bar{\mu} = \prod_{k\in N} \mu_k$*.*

(b) $(S_i, \prod_{j\in N\setminus\{i\}} S_j, u_i)$ *is player i's decision problem in* $\Gamma_{ac}(m)$ *relative to the probability spaces* $\{(C_j, \mathcal{C}_j, \mu_j) \mid j \in N\}$*.*

Each S_j embodies three restrictions. First, it is defined relative to the exogenous measure μ_j. Secondly, as $S_j \subset L^p_{C_j} \subset L^1_{C_j}$ and $L^p_{C_j} \neq L^1_{C_j}$, S_j cannot accommodate all the densities relative to μ_j. Thirdly, S_j is bounded relative to the norm $\|.\|_{p,C_j}$.

The players in $\Gamma_{ac}(m)$ randomise independently over their pure strategies by choosing densities. If they choose the profile of densities s, then $u_i(s) = \int_C \bar{\mu}(dc)\, [\sum_{j\in N\setminus\{i\}} w_j \circ \pi_{i,j}](c)\bar{s}(c) = \sum_{j\in N\setminus\{i\}} \int_C \bar{\mu}(dc)\, w_j \circ \pi_{i,j}(c)\bar{s}(c)$. As $\int_{C_k} \mu_k(dc_k)\, s_k(c_k) = 1$ for every $k \in N$, we have $\int_C \bar{\mu}(dc)\, w_j \circ \pi_{i,j}(c)\bar{s}(c) = \int_{C_i\times C_j} \mu_i \times \mu_j(dc_i, dc_j)\, w_j(c_i, c_j)s_i(c_i)s_j(c_j) \equiv \Phi_j(s_i, s_j)$ for every $j \in N\setminus\{i\}$. Hence, $u_i = \sum_{j\in N\setminus\{i\}} \Phi_j \circ \pi_{i,j}$.

Clearly, $\varphi_j(c_i, s_j) = \int_{C_j} \mu_j(dc_j)\, w_j(c_i, c_j)s_j(c_j)$ represents player i's expected payoff from 'the interaction with player j' when i plays $c_i \in C_i$ and j randomises using $s_j \in S_j$. By Lemma 5.8, $\varphi_j(c_i, .)$ is continuous at s_j for every c_i and every s_j. For every $s_j \in S_j$, the following result assumes that the continuity of $\varphi_j(c_i, .)$ at s_j is uniform across $c_i \in C_i$.

Theorem 5.10 *Consider a game* $\Gamma = \{N, (C_j, v_j)_{j\in N}\}$ *and player i's decision problem* $(S_i, \prod_{j\in N\setminus\{i\}} S_j, u_i)$ *as per Definition 5.9. If, for every* $j \in N \setminus \{i\}$*, the family of functions* $\{\varphi_j(c_i, .) \mid c_i \in C_i\}$ *is pointwise equicontinuous when* $L^p_{C_j}$ *is given the weak topology, then* $s_i \in S_i$ *is strongly dominated if and only if it is not a best reply.*

Proof. Give $L^p_{C_j}$ its weak topology for every $j \in N$ and $L^p_{C_i\times C_j}$ its weak topology for every $j \in N \setminus \{i\}$. The result follows from Theorem 5.1 if its assumptions are satisfied. We verify that this is indeed the case.

Assumption (a) is satisfied by Definition 5.9. Lemma 5.7 implies that assumptions (b) and (c) are satisfied.

Lemma 5.8 implies that Φ_j is real-valued and continuous for every $j \in N \setminus \{i\}$. Since the projection $\pi_{i,j} : S \to S_i \times S_j$ is continuous, $\Phi_j \circ \pi_{i,j}$ is continuous for every $j \in N \setminus \{i\}$. It follows that $u_i = \sum_{j \in N \setminus \{i\}} \Phi_j \circ \pi_{i,j}$ is real-valued and continuous. So, assumption (d) is satisfied.

Consider $t \in (0,1)$, $s_i \in S_i$, and $s_{-i}, s'_{-i} \in S_{-i}$. Applying Lemma 5.8, $\Phi_j(s_i, ts_j + (1-t)s'_j) = t\Phi_j(s_i, s_j) + (1-t)\Phi_j(s_i, s'_j)$ for every $j \in N \setminus \{i\}$. Summing both sides over $j \in N \setminus \{i\}$ yields $u_i(s_i, ts_{-i} + (1-t)s'_{-i}) = \sum_{j \in N \setminus \{i\}} \Phi_j \circ \pi_{i,j}(s_i, ts_{-i} + (1-t)s'_{-i}) = \sum_{j \in N \setminus \{i\}} \Phi_j(s_i, ts_j + (1-t)s'_j) = \sum_{j \in N \setminus \{i\}} [t\Phi_j(s_i, s_j) + (1-t)\Phi_j(s_i, s'_j)] = t \sum_{j \in N \setminus \{i\}} \Phi_j \circ \pi_{i,j}(s_i, s_{-i}) + (1-t) \sum_{j \in N \setminus \{i\}} \Phi_j \circ \pi_{i,j}(s_i, s'_{-i}) = tu_i(s_i, s_{-i}) + (1-t)u_i(s_i, s'_{-i})$. So, assumption (e) is satisfied. Satisfaction of assumption (f) is verified analogously. ■

5.4 Duality I and additive mixed extensions

Consider a game $\Gamma = \{N, (C_j, v_j)_{j \in N}\}$ wherein N is a finite set of players, C_j is player j's set of pure strategies and, setting $C = \prod_{j \in N} C_j$, $v_j : C \to \Re$ is her utility function. We shall supplement Γ with structure that enables the construction of $\Gamma(m)$ as Γ's additive mixed extension; the reader is referred to Section 3 for the necessary technical background. Then we shall apply Duality I to player i's decision problem in $\Gamma(m)$.

Suppose $\mathcal{C}_j$ is an algebra on C_j for every $j \in N$. Let $\mathcal{E}$ be the collection of rectangles $\prod_{j \in N} E_j$, where $E_j \in \mathcal{C}_j$ for every $j \in N$. Then, $\mathcal{E}$ is a semialgebra on C.[6] Let $\mathcal{C}$ be the algebra on C generated by $\mathcal{E}$. By Lemma 3.5, $\mathcal{C}$ consists of finite unions of pairwise disjoint rectangles in $\mathcal{E}$.

Let $j \in N \setminus \{i\}$. Let $\mathcal{E}_{i,j}$ be the collection of rectangles $E \times F$, where $E \in \mathcal{C}_i$ and $F \in \mathcal{C}_j$. By specialising the argument for $\mathcal{E}$, $\mathcal{E}_{i,j}$ is a semialgebra on $C_i \times C_j$. Let $\mathcal{C}_i \times \mathcal{C}_j$ be the algebra generated on $C_i \times C_j$ by $\mathcal{E}_{i,j}$. By Lemma 3.5, $\mathcal{C}_i \times \mathcal{C}_j$ consists of finite unions of pairwise disjoint rectangles in $\mathcal{E}_{i,j}$.

Given a profile of additive measures $(x_j)_{j \in N} \in \prod_{j \in N} \text{ba}(C_j, \mathcal{C}_j)$, there exists a unique product measure $\bar{x} \in \text{ba}(C, \mathcal{C})$ such that $\bar{x}(\prod_{j \in N} E_j) = \prod_{j \in N} x_j(E_j)$ for every rectangle $\prod_{j \in N} E_j \in \mathcal{E}$ (Dunford and Schwartz [7], Lemma III.11.1). With this preparation, we have the following definition.

Definition 5.11 *Consider* $\Gamma = \{N, (C_j, v_j)_{j \in N}\}$, $\{\mathcal{C}_j \mid j \in N\}$, *and* $i \in N$ *such that:* N *is finite;* $\mathcal{C}_j$ *is an algebra on* C_j *for every* $j \in N$*; and* $v_i =$

[6] Clearly, $C, \emptyset \in \mathcal{E}$. Let $N = \{1, \ldots, n\}$, $E = \prod_{j=1}^n E_j \in \mathcal{E}$, and $F = \prod_{j=1}^n F_j \in \mathcal{E}$. As $E_j \cap F_j \in \mathcal{C}_j$ for $j \in N$, $E \cap F = \prod_{j=1}^n (E_j \cap F_j) \in \mathcal{E}$. Finally, suppose $E \subset F$. Let $A_0 = E$, $A_i = \prod_{j=1}^i F_j \times \prod_{j=i+1}^n E_j$ for $i = 1, \ldots, n-1$, and $A_n = F$. Then, $A_i \in \mathcal{E}$ for $i = 0, \ldots, n$, $E = A_0 \subset A_1 \subset \ldots \subset A_n = F$, $A_1 \setminus A_0 = (F_1 \setminus E_1) \times \prod_{j=2}^n E_j \in \mathcal{E}$, $A_i \setminus A_{i-1} = \prod_{j=1}^{i-1} F_j \times (F_i \setminus E_i) \times \prod_{j=i+1}^n E_j \in \mathcal{E}$ for $i = 2, \ldots, n-1$, and $A_n \setminus A_{n-1} = \prod_{j=1}^{n-1} F_j \times (F_n \setminus E_n) \in \mathcal{E}$, as $F_i \setminus E_i \in \mathcal{C}_i$ for $i = 1, \ldots, n$. Hence, $\mathcal{E}$ is a semialgebra.

$\sum_{j \in N\setminus\{i\}} w_j \circ \pi_{i,j}$, *where* $\pi_{i,j}$ *projects* C *on* $C_i \times C_j$ *and* $w_j \in B(C_i \times C_j, \mathcal{C}_i \times \mathcal{C}_j)$ *for every* $j \in N \setminus \{i\}$.

(a) $\Gamma(m) = \{N, (\Delta(\mathcal{C}_j), V_j)_{j\in N}\}$ *is called the additive mixed extension of* Γ *wherein, for every* $j \in N$, $\Delta(\mathcal{C}_j)$ *is the set of additive probability measures on* $\mathcal{C}_j$ *and* $V_j : \prod_{k\in N} \Delta(\mathcal{C}_k) \to \Re$ *is given by* $V_j(x) = \int_C \bar{x}(dc)\, v_j(c)$ *for the product measure* $\bar{x} \in \Delta(\mathcal{C})$ *resulting from the profile* $x = (x_k)_{k\in N} \in \prod_{k\in N} \Delta(\mathcal{C}_k)$.[7]

(b) $(\Delta(\mathcal{C}_i), \prod_{j\in N\setminus\{i\}} \Delta(\mathcal{C}_j), V_i)$ *is player i's decision problem in* $\Gamma(m)$.

In order to analyse i's decision problem, we first note some facts regarding the relationship between product measures and component measures.

Lemma 5.12 *Consider* Γ, $\{\mathcal{C}_j \mid j \in N\}$, *and* $i \in N$ *as per Definition 5.11.*

(A) *Suppose we give* $\mathrm{ba}(C_j, \mathcal{C}_j)$ *its weak* topology for every* $j \in N$, $\mathrm{ba}(C, \mathcal{C})$ *its weak* topology, and* $\prod_{j\in N} \mathrm{ba}(C_j, \mathcal{C}_j)$ *the product topology. Then, the mapping* $\prod_{j\in N} \Delta(\mathcal{C}_j) \ni (x_j)_{j\in N} \mapsto \bar{x} \in \Delta(\mathcal{C})$ *is continuous.*

(B) $w_j \circ \pi_{i,j} \in B(C, \mathcal{C})$ *for every* $j \in N \setminus \{i\}$.

(C) *Given* $(x_k)_{k\in N} \in \prod_{k\in N} \Delta(\mathcal{C}_k)$ *and* $\bar{x} \in \Delta(\mathcal{C})$, $\int_C \bar{x}(dc)\, w_j \circ \pi_{i,j}(c) = \int_{C_i\times C_j} \bar{x} \circ \pi_{i,j}^{-1}(dc_i, dc_j)\, w_j(c_i, c_j)$ *for every* $j \in N \setminus \{i\}$.

Next, we derive the properties of V_i (without assuming v_i's continuity).

Lemma 5.13 *Consider* Γ, $\{\mathcal{C}_j \mid j \in N\}$, *and* $i \in N$ *as per Definition 5.11.*

(A) *If the hypotheses of Lemma 5.12(A) are satisfied, then* V_i *is continuous.*

(B) $V_i(x, ty + (1-t)z) = tV_i(x, y) + (1-t)V_i(x, z)$ *for* $x \in \Delta(\mathcal{C}_i)$, $y, z \in \prod_{j\in N\setminus\{i\}} \Delta(\mathcal{C}_j)$, *and* $t \in (0, 1)$.

(C) $V_i(tx + (1-t)y, z) = tV_i(x, z) + (1-t)V_i(y, z)$ *for* $x, y \in \Delta(\mathcal{C}_i)$, $z \in \prod_{j\in N\setminus\{i\}} \Delta(\mathcal{C}_j)$, *and* $t \in (0, 1)$.

Finally, we can apply Duality I to player i's decision problem in $\Gamma(m)$.

Theorem 5.14 *Consider* Γ, $i \in N$, $\{\mathcal{C}_j \mid j \in N\}$, *and player i's decision problem* $(\Delta(\mathcal{C}_i), \prod_{j\in N\setminus\{i\}} \Delta(\mathcal{C}_j), V_i)$ *as per Definition 5.11. If the hypotheses of Lemma 5.12(A) are satisfied, then* $x_0 \in \Delta(\mathcal{C}_i)$ *is strongly dominated if and only if it is not a best reply.*

Proof. It suffices to verify that $\{N, (\Delta(\mathcal{C}_j), V_j)_{j\in N}\}$ satisfies Theorem 5.1's hypotheses. Hypothesis (a) is satisfied by assumption. As *per* Section 3, $\Delta(\mathcal{C}_i)$ satisfies hypothesis (b) and $\Delta(\mathcal{C}_j)$ satisfies hypothesis (c) for $j \in N \setminus \{i\}$. By Lemma 5.13, V_i satisfies hypotheses (d), (e), and (f). ■

[7] Although V_j is defined formally for every $j \in N$ as a part of $\Gamma(m)$'s definition, our results rely only on V_i and are independent of V_j for $j \in N \setminus \{i\}$.

5.5 Duality II and σ-additive mixed extensions

Consider $(\Delta(C_i), \prod_{j \in N\setminus\{i\}} \Delta(C_j), V_i)$ as *per* Definition 5.5. In order to characterise weak dominance in terms of internal-best replies for this problem, we shall apply Theorem 2.3. Accordingly, we henceforth set

$$(X, Y, u) \equiv \left(\Delta(C_i), \prod_{j \in N\setminus\{i\}} \Delta(C_j), V_i \right) \tag{2}$$

Application of Theorem 2.3 to this (X, Y, u) requires $Y^* \neq \emptyset$. Moreover, Theorem 2.3(A) applies only to $x_0 \in \operatorname{Int} X$. Unfortunately, $\operatorname{Int} X = \emptyset$ and $Y^* = \emptyset$ even in the simplest version of (X, Y, u) given by Equation (2).[8] However, the following transformation of (X, Y, u) is the first step towards resolving these issues.

Definition 5.15 *Consider (X, Y, u) as defined by Equation (2) and Definition 5.5, $\alpha \in X = \Delta(C_i)$, and $\beta \in Y = \prod_{j \in N\setminus\{i\}} \Delta(C_j)$. Player i's transformed problem is (K, L, w), where $K = X - \{\alpha\} = \Delta(C_i) - \{\alpha\}$, $L = Y - \{\beta\} = \prod_{j \in N\setminus\{i\}}[\Delta(C_j) - \{\beta_j\}]$, and $w : K \times L \to \Re$ is given by $w(x, y) = u(x + \alpha, y + \beta) = V_i(x + \alpha, y + \beta)$ for $(x, y) \in K \times L$.*

It is easily verified that $x_0 \in X$ is weakly dominated with respect to (X, Y, u) if and only if $x_0 - \alpha$ is weakly dominated with respect to (K, L, w). The second step towards our objective is to define a transformed notion of an internal-best reply.

Definition 5.16 *Consider (X, Y, u) as per Equation (2) and Definition 5.5, $\alpha \in X$, $\beta \in Y$, and (K, L, w) given by Definition 5.15. Then, $x_0 \in X$ is said to be an affine internal-best reply with respect to (X, Y, u) if $x_0 - \alpha$ is an internal-best reply with respect to (K, L, w).*

Using Definitions 5.15 and 5.16, the following propositions are equivalent:

- *$x_0 \in K$ is weakly dominated with respect to (K, L, w) if and only if it is not an internal-best reply with respect to (K, L, w).*
- *$x_0 + \alpha \in X$ is weakly dominated with respect to (X, Y, u) if and only if it is not an affine internal-best reply with respect to (X, Y, u).*

We validate the former proposition by applying Theorem 2.3 to (K, L, w), thereby also validating the latter one. The steps for doing this are as follows:

[8] In order to illustrate the problem and its proposed solution, consider the game $\Gamma = \{\{1, 2\}, (C_1, v_1), (C_2, v_2)\}$ with $|C_1| = |C_2| = 2$. Set $i = 1$, $X = \Delta(C_1)$, and $Y = \Delta(C_2)$. $\Re^2$ is the linear span of X and Y, and $\operatorname{Int} X = \emptyset$ and $Y^* = \emptyset$ with respect to $\Re^2$. But, if we set $K = X - \{(1, 0)\}$ and $L = Y - \{(1, 0)\}$, then the linear spans of these sets are $\mathcal{K} = \{p \in \Re^2 \mid p_1 + p_2 = 0\}$ and $\mathcal{L} = \{p \in \Re^2 \mid p_1 + p_2 = 0\}$ respectively. Consequently, $\operatorname{Int} K \neq \emptyset$ with respect to $\mathcal{K}$ and $L^* \neq \emptyset$ with respect to $\mathcal{L}$.

I. Lemma 5.17 will facilitate the embedding of K and L in TVSs that meet the requirements of Theorem 2.3 and ensure that $\operatorname{Int} K \neq \emptyset$ and $\operatorname{Int} L \neq \emptyset$. $\operatorname{Int} K \neq \emptyset$ will ensure that the assumption $x_0 \in \operatorname{Int} K$ in Theorem 5.18 is non-vacuous. $\operatorname{Int} L \neq \emptyset$ will ensure $L^* \neq \emptyset$, thereby allowing the application of Theorem 2.3(A) to (K, L, w) and $x_0 \in \operatorname{Int} K$.

II. Given the hypotheses and proof of Theorem 5.18, the application of Theorem 2.3(B) to (K, L, w) and $x_0 \in \operatorname{Fr} K$ only requires that Theorem 2.3(B)'s supplementary hypothesis is satisfied. In the general case specified by Definition 5.5, we can show this for points in $\operatorname{Fr} K$ that satisfy a relative interiority condition. This condition is not vacuous as Lemma 5.20 shows that there are points in $\operatorname{Fr} K$ that do meet the relative interiority condition in the general setting given by Definition 5.15. For such $x_0 \in \operatorname{Fr} K$, we can apply Theorem 2.3(B) to deduce Theorem 5.21. Extendability of the weak dominance duality result to all of $\operatorname{Fr} K$ in the setting of Definition 5.15 is an open question.

III. However, if (K, L, w) is generated by a finite game *via* Definitions 5.5 and 5.15, then it will be shown in the proof of Theorem 5.22 that Theorem 2.3(B)'s supplementary hypothesis is satisfied for every $x_0 \in \operatorname{Fr} K$, thereby allowing the application of Theorem 2.3 to every $x_0 \in K$.

As motivation for step I of our programme, consider a set T with $|T| = n \in \mathcal{N}$. Given $\alpha \in \Delta(T)$, it is evident that $\mathcal{S} = \{x \in \Re^n \mid \sum_{i=1}^{n} x_i = 1\} - \{\alpha\}$ is the linear span of $\Delta(T) - \{\alpha\}$ and $\operatorname{Int}[\Delta(T) - \{\alpha\}] \neq \emptyset$ relative to $\mathcal{S}$. These observations generalise as follows.

Lemma 5.17 *Suppose T is a compact metric space, $\alpha \in \Delta(T)$, $\operatorname{rca}(T)$ is given its weak* topology, and $\mathcal{S}$ is the closed linear span of $\Delta(T) - \{\alpha\}$ in $\operatorname{rca}(T)$. Then,*

(A) $\mathcal{S}$ is a Hausdorff LCS, and

(B) $\operatorname{Int}[\Delta(T) - \{\alpha\}] \neq \emptyset$ relative to $\mathcal{S}$.

The following result is step I of the programme.

Theorem 5.18 *Consider (K, L, w) given by Definition 5.15.*

(A) If $\operatorname{rca}(C_i)$ is given its weak topology, then $\operatorname{Int} K \neq \emptyset$ relative to the closed linear span of K in $\operatorname{rca}(C_i)$.*

(B) Suppose $\operatorname{rca}(C_j)$ is given its weak topology for every $j \in N$. If $x_0 \in \operatorname{Int} K$ relative to the closed linear span of K in $\operatorname{rca}(C_i)$ and $w(x_0, .) \neq w(x, .)$ for every $x \in K \setminus \{x_0\}$, then x_0 is weakly dominated if and only if it is not an internal-best reply.*

Proof. Consider (K, L, w) *ex hypothesi.*

(A) Let $\mathcal{K}$ be the closed linear span of $K = \Delta(C_i) - \{\alpha\}$ in $\text{rca}(C_i)$. By Lemma 5.17, $\mathcal{K}$ is a Hausdorff LCS and $\text{Int}\, K \neq \emptyset$ relative to $\mathcal{K}$.

(B) For $j \in N \setminus \{i\}$, let $\mathcal{L}_j$ be the closed linear span of $L_j = \Delta(C_j) - \{\beta_j\}$ in $\text{rca}(C_j)$. By Lemma 5.17, $\mathcal{L}_j$ is a Hausdorff LCS. Therefore, so is $\mathcal{L} = \prod_{j \in N\setminus\{i\}} \mathcal{L}_j$.

Lemma 5.17 also implies $\text{Int}\, L_j \neq \emptyset$ relative to $\mathcal{L}_j$. As N is finite, $\text{Int}\, L \neq \emptyset$ relative to $\mathcal{L}$. Therefore, $L^* = \text{Int}\, L \neq \emptyset$ (Dunford and Schwartz [7], Theorem V.2.1).

Using the properties of $\{\Delta(C_j) \mid j \in N\}$ derived in the proof of Theorem 5.6, K (resp., L) is a convex, compact, and metrisable subset of $\mathcal{K}$ (resp., $\mathcal{L}$). So, K and L satisfy hypotheses (a) and (b) of Theorem 2.3.

Hypothesis (c) of Theorem 2.3 is satisfied by w because the product topology on $K \times L$ makes projections continuous, translations in a TVS are continuous, and V_i is continuous by the proof of Theorem 5.6.

As the proof of Theorem 5.6 shows that V_i satisfies hypotheses (d) and (e) of Theorem 2.3, it is easily confirmed that so does w. The result follows from Theorem 2.3(A). ∎

Next, we proceed to step II of the programme outlined above. In order to formulate the relative interiority property, we need some technical facts. For K given by Definition 5.15, we represent K and $\text{Fr}\, K$ by the formulae

$$K = \cap_{f \in \mathcal{F}} H_f^- \quad \text{and} \quad \text{Fr}\, K = \cup_{f \in \mathcal{F}} (K \cap H_f) = \cup_{f \in \mathcal{F}} (K \cap A_f) \qquad (3)$$

which we interpret and derive as follows:

- With $\text{rca}(C_i)$ given its weak* topology, let $\mathcal{K}$ be the closed linear span of K in $\text{rca}(C_i)$. By Lemma 5.17, $\mathcal{K}$ is a Hausdorff LCS. Let $\mathcal{F}$ be the family of continuous non-zero linear functionals $f : \mathcal{K} \to \Re$. Each $f \in \mathcal{F}$ generates a *supporting hyperplane* of K, given by $H_f = \{x \in \mathcal{K} \mid f(x) = \max f(K)\}$.[9] Then, $K = \cap_{f \in \mathcal{F}} H_f^-$, where $H_f^- = \{x \in \mathcal{K} \mid f(x) \leq \max f(K)\}$ (Schaefer [12], Theorem II.10.1) is called a *closed supporting halfspace.*
- Since $\text{Int}\, K \neq \emptyset$ relative to $\mathcal{K}$ by Lemma 5.17, K is a *convex body* (Schaefer [12], p. 40). It follows that $\text{Fr}\, K \subset \cup_{f \in \mathcal{F}} H_f$ (Schaefer [12], Corollary II.9.1). As K is closed in $\mathcal{K}$, we have $\text{Fr}\, K \subset K \cap (\cup_{f \in \mathcal{F}} H_f) =$

[9] A supporting hyperplane for K is a maximal proper affine subspace of $\mathcal{K}$ (i.e., with codimension 1) that is closed in $\mathcal{K}$, intersects K, and K is contained in one of the closed halfspaces generated by it (Schaefer [12], Sections I.4 and II.9). The other supporting hyperplane associated with f is $H_{-f} = \{x \in \mathcal{K} \mid -f(x) = \max(-f)(K)\} = \{x \in \mathcal{K} \mid f(x) = \min f(K)\}$.

$\cup_{f\in\mathcal{F}}(K\cap H_f)\subset \operatorname{Fr} K$ (Schaefer [12], Lemma II.9.1). So, $\operatorname{Fr} K = \cup_{f\in\mathcal{F}}(K\cap H_f)$, where $K\cap H_f$ is called the *facet of* K *generated by* f.

- Let A_f be the *closed affine span* of $K\cap H_f$ in $\mathcal{K}$. Evidently, $K\cap H_f \subset K\cap A_f$. As H_f is a closed affine subspace containing $K\cap H_f$, we have $A_f\subset H_f$. So, $K\cap H_f\supset K\cap A_f$. Therefore, $K\cap H_f = K\cap A_f$ and $\operatorname{Fr} K = \cup_{f\in\mathcal{F}}(K\cap A_f)$.

The following example illustrates the notions of a supporting hyperplane H_f, a closed affine span A_f, and the interior of facet $K\cap H_f = K\cap A_f$ relative to H_f (resp., A_f, $\operatorname{Fr} K$). It also prefigures Lemma 5.20.

Example 5.19 *Consider* C_i *with* $|C_i| = 3$. *Then,* $\Delta(C_i)$ *is the convex hull of the extreme points* $e^1 = (1, 0, 0)$, $e^2 = (0, 1, 0)$, *and* $e^3 = (0, 0, 1)$. *Given* $\alpha = (1/3, 1/3, 1/3) \in \Delta(C_i)$, *let* $K = \Delta(C_i) - \{\alpha\} = \operatorname{co}\{e^1-\alpha, e^2-\alpha, e^3-\alpha\}$. $\Delta(C_i)$ *and* K *are convex and compact. The linear span of* K *is a hyperplane in* $\Re^3$, *namely the 2-dimensional subspace* $\mathcal{K} = \{p\in\Re^3 \mid \sum_{i=1}^3 p_i = 0\}$.

Consider $x_0 = e^1-\alpha \in \operatorname{Fr} K$, *which is an extreme point of* K. *A supporting hyperplane of* K *at* x_0, *relative to* $\mathcal{K}$, *is the 1-dimensional affine subspace* $H_0 = \{x_0 + t(e^3 - e^2) \mid t\in\Re\}$; H_0 *is but one of an infinite family of supporting hyperplanes of* K *at* x_0. *Clearly,* $K\cap H_0 = \{x_0\}$ *and the affine span of* $K\cap H_0$ *is the 0-dimensional affine space* $A_0 = \{x_0\}$. *Observe that* $K\cap H_0 = K\cap A_0 = \{x_0\}$, $A_0\subset H_0$, $A_0\neq H_0$, *and*

$$\operatorname{Int}(K\cap H_0) = \operatorname{Int}(K\cap A_0) = \begin{cases}\emptyset, & \textit{relative to } H_0\\ \{x_0\}, & \textit{relative to } A_0.\end{cases}$$

Now consider $x_1 = (e^1+e^3)/2-\alpha \in \operatorname{Fr} K$. *The unique supporting hyperplane of* K *at* x_1, *relative to* $\mathcal{K}$, *is the 1-dimensional affine subspace* $H_1 = \{x_1 + t(e^3-e^1) \mid t\in\Re\}$. *Then,* $K\cap H_1 = \{x_1 + t(e^3-e^1) \mid t \in [-1/2, 1/2]\}$ *is a convex and compact facet of* K *and the affine span of* $K\cap H_1$ *is* $A_1 = H_1$. *So,* $K\cap H_1 = K\cap A_1$ *and* $x_1 \in \operatorname{Int}(K\cap H_1) = \operatorname{Int}(K\cap A_1) = \{x_1 + t(e^3-e^1) \mid t\in(-1/2, 1/2)\}$ *relative to* $H_1 = A_1$.

Note that H_1 *is also a supporting hyperplane of* K *at* x_0, *relative to* $\mathcal{K}$. *However, unlike* x_1, $x_0\notin \operatorname{Int}(K\cap H_1) = \operatorname{Int}(K\cap A_1)$ *relative to* $H_1 = A_1$.

Let $H_2 = \{e^2-\alpha+t(e^1-e^2) \mid t\in\Re\}$ *and* $H_3 = \{e^3-\alpha+t(e^2-e^3) \mid t\in\Re\}$. *Like* H_1, H_2 *and* H_3 *are 1-dimensional supporting hyperplanes of* K *relative to* $\mathcal{K}$, *which generate the facets* $K\cap H_2 = \{e^2-\alpha+t(e^1-e^2) \mid t\in[0,1]\}$ *and* $K\cap H_3 = \{e^3-\alpha+t(e^2-e^3) \mid t\in[0,1]\}$. *Then,* $\operatorname{Fr} K = \cup_{i=1}^3(K\cap H_i)$, *i.e., the frontier of* K *relative to* $\mathcal{K}$ *is the union of convex and compact facets. Moreover,* $x_1\in\operatorname{Int}(K\cap H_1) = \operatorname{Int}(K\cap A_1)$ *relative to* $\operatorname{Fr} K$, *i.e., there is a set* V *that is open in* $\mathcal{K}$ *such that* $x_1\in V\cap\operatorname{Fr} K\subset K\cap H_1 = K\cap A_1$.

As suggested by this example, we show more generally that a facet of K has a nonempty interior relative to its closed affine span. Moreover, if a frontier point of K belongs to a facet's interior relative to a supporting hyperplane, then it is in the facet's interior relative to the frontier of K.

Lemma 5.20 *Consider K given by Definition 5.15. Suppose* $\mathrm{rca}(C_i)$ *is given its weak* topology, $\mathcal{K}$ is the closed linear span of K in* $\mathrm{rca}(C_i)$*, and $f : \mathcal{K} \to \Re$ is a continuous non-zero linear functional.*

(A) $K \cap H_f$ is convex, compact, metrisable, and $\mathrm{Int}\,(K \cap H_f) \neq \emptyset$ *relative to A_f.*

(B) If $x_0 \in \mathrm{Int}\,(K \cap H_f)$ relative to A_f and $H_f = A_f$, then $x_0 \in \mathrm{Int}\,(K \cap H_f)$ relative to $\mathrm{Fr}\,K$*.*

We can now verify the supplementary hypothesis of Theorem 2.3(B) for points in $\mathrm{Fr}\,K$ that satisfy the requirements of Lemma 5.20(B).

Theorem 5.21 *Suppose (K, L, w) is given by Definition 5.15,* $\mathrm{rca}(C_j)$ *is given its weak* topology for every $j \in N$, $\mathcal{K}$ is the closed linear span of K in* $\mathrm{rca}(C_i)$*, and $f : \mathcal{K} \to \Re$ is a continuous non-zero linear functional.*

If $x_0 \in \mathrm{Int}\,(K \cap H_f)$ relative to H_f, $H_f = A_f$, and $w(x_0, .) \neq w(x, .)$ for every $x \in K \setminus \{x_0\}$, then x_0 is weakly dominated if and only if it is not an internal-best reply.

Proof. Given the assumptions, it suffices to copy the proof of Theorem 5.18 to verify that all the hypotheses of Theorem 2.3 are satisfied. As Equation (3) implies that $x_0 \in K \cap H_f \subset \mathrm{Fr}\,K$, it only remains to verify that the supplementary hypothesis of Theorem 2.3(B) is satisfied. Consider $x \in K \setminus \{x_0\}$.

1. Let $\mathcal{T}$ denote the topology of $\mathcal{K}$. As $x_0 \in \mathrm{Int}\,(K \cap H_f)$ relative to H_f, there is $V_1 \in \mathcal{T}$ such that $x_0 \in V_1 \cap H_f \subset K \cap H_f$. By Lemma 5.20(B), $x_0 \in \mathrm{Int}\,(K \cap H_f)$ relative to $\mathrm{Fr}\,K$. So, there is $V_2 \in \mathcal{T}$ such that $x_0 \in V_2 \cap \mathrm{Fr}\,K \subset K \cap H_f$. Let $V = V_1 \cap V_2$ and $C = \mathrm{Fr}\,K \setminus V$. Then, $V \in \mathcal{T}$, $x_0 \notin C$, and as $\mathrm{Fr}\,K$ is compact, C is compact.

   Note that $V \cap \mathrm{Fr}\,K \subset V_2 \cap \mathrm{Fr}\,K \subset K \cap H_f$. Consider $z \in \mathrm{Fr}\,K \setminus H_f$. If $z \in V$, then $z \in V \cap \mathrm{Fr}\,K \subset H_f$, a contradiction. So, $\mathrm{Fr}\,K \setminus H_f \subset \mathrm{Fr}\,K \setminus V = C$.

2. Let $h(t) = x_0 + t(x - x_0)$ for $t \geq 1$. Then, $h : [1, \infty) \to \mathcal{K}$ is continuous.

3. As $\mathcal{K}$ is a Hausdorff LCS by Lemma 5.17, $\mathcal{T}$ is generated by a total family of continuous seminorms $\mathcal{P}$. As $\mathcal{P}$ is total and $x_0 \neq x$, there exists $p \in \mathcal{P}$ such that $p(x_0 - x) > 0$. Since $p(x_0 - h(t)) = p(t(x_0 - x)) = tp(x_0 - x)$ for $t \geq 1$, it follows that $\{p(x_0 - y) \mid y \in h([1, \infty))\}$ is unbounded above. As p is continuous and K is compact, $\{p(x_0 - y) \mid y \in K\}$ is bounded. So, $h(t_0) \notin K$ for some $t_0 > 1$.

4. Consider $x \notin H_f$.

   (a) If $h(t) \in H_f$ for some $t \geq 1$, then $x = t^{-1}h(t) + (1-t^{-1})x_0 \in H_f$, which is a contradiction. So, $h(t) \notin H_f$ for every $t \geq 1$.

   (b) Suppose $h^{-1}(\operatorname{Fr} K) = \emptyset$. Then, $h(1) = x \in K \setminus \operatorname{Fr} K = \operatorname{Int} K$. So, $h^{-1}(\operatorname{Int} K) \neq \emptyset$. By step 3, $h^{-1}(\mathcal{K} \setminus K) \neq \emptyset$. By definition, $h^{-1}(\operatorname{Int} K) \cap h^{-1}(\mathcal{K} \setminus K) = \emptyset$ and $[1, \infty) \supset h^{-1}(\operatorname{Int} K) \cup h^{-1}(\mathcal{K} \setminus K)$. Consider $t \in [1, \infty)$ such that $t \notin h^{-1}(\operatorname{Int} K)$. Since $h^{-1}(\operatorname{Fr} K) = \emptyset$, we have $t \notin h^{-1}(\operatorname{Int} K) \cup h^{-1}(\operatorname{Fr} K) = h^{-1}(\operatorname{Int} K \cup \operatorname{Fr} K) = h^{-1}(K)$. So, $t \in h^{-1}(\mathcal{K} \setminus K)$. Thus, $[1, \infty) = h^{-1}(\operatorname{Int} K) \cup h^{-1}(\mathcal{K} \setminus K)$. Since $\operatorname{Int} K \in \mathcal{T}$, $\mathcal{K} \setminus K \in \mathcal{T}$, and h is continuous, $h^{-1}(\operatorname{Int} K)$ and $h^{-1}(\mathcal{K} \setminus K)$ are open in $[1, \infty)$. This implies $[1, \infty)$ is topologically disconnected, which is a contradiction.

   (c) Thus, $h(t_1) \in \operatorname{Fr} K$ for some $t_1 \in [1, \infty)$. Using step (a) and step 1, we have $h(t_1) \in \operatorname{Fr} K \setminus H_f \subset C$.

5. Consider $x \in H_f$.

   (a) As $x_0 \in H_f$ and H_f is an affine space, $h(t) \in H_f$ for every $t \geq 1$.

   (b) Suppose there exists $t \geq 1$ such that $h(t) \in \operatorname{Fr}(K \cap H_f)$ relative to H_f. Then, $h(t) \in K \cap H_f \subset \operatorname{Fr} K$ and $h(t) \notin \operatorname{Int}(K \cap H_f)$ relative to H_f. As $V_1 \cap H_f \subset K \cap H_f$, $h(t) \in V_1 \cap H_f$ implies $h(t) \in \operatorname{Int}(K \cap H_f)$ relative to H_f, which is a contradiction. So, $h(t) \notin V_1 \cap H_f$. Since $h(t) \in H_f$, this means $h(t) \notin V_1$. So, $h(t) \in \operatorname{Fr} K \setminus V_1 \subset \operatorname{Fr} K \setminus V = C$, as required.

   (c) Suppose there is no $t \geq 1$ such that $h(t) \in \operatorname{Fr}(K \cap H_f)$ relative to H_f.

   Since $h(1) = x \in K \cap H_f$ and $h(1) \notin \operatorname{Fr}(K \cap H_f)$ relative to H_f, we have $h(1) \in \operatorname{Int}(K \cap H_f)$ relative to H_f. As h is continuous and $\operatorname{Int}(K \cap H_f)$ is open relative to H_f, $h^{-1}(\operatorname{Int}(K \cap H_f))$ is open in $[1, \infty)$ and $1 \in h^{-1}(\operatorname{Int}(K \cap H_f))$.

   By step 3, $h(t_0) \in H_f \setminus K$. Since h is continuous and $H_f \setminus K$ is open in H_f, $h^{-1}(H_f \setminus K)$ is open in $[1, \infty)$ and $t_0 \in h^{-1}(H_f \setminus K)$. If there is some $t \in [1, \infty)$ such that $h(t) \in \operatorname{Int}(K \cap H_f) \subset K \cap H_f$ and $h(t) \in H_f \setminus K$, we have a contradiction. So, $h^{-1}(\operatorname{Int}(K \cap H_f)) \cap h^{-1}(H_f \setminus K) = \emptyset$.

   Consider $t \geq 1$ such that $h(t) \notin \operatorname{Int}(K \cap H_f)$ relative to H_f. Using this step's hypothesis, $h(t) \notin \operatorname{Int}(K \cap H_f) \cup \operatorname{Fr}(K \cap H_f) = K \cap H_f$. Since $h(t) \in H_f$, this implies $h(t) \notin K$, and so, $h(t) \in H_f \setminus K$. Thus, $h^{-1}(\operatorname{Int}(K \cap H_f)) \cup h^{-1}(H_f \setminus K) = [1, \infty)$.

   These arguments imply $[1, \infty)$ is topologically disconnected, which is a contradiction.

(d) Thus, there exists $t \geq 1$ such that $h(t) \in \mathrm{Fr}\,(K \cap H_f)$ relative to H_f. Hence, $h(t) \in \mathrm{Fr}\,(K \cap H_f) \subset K \cap H_f \subset \mathrm{Fr}\,K$ and $h(t) \notin \mathrm{Int}\,(K \cap H_f)$ relative to H_f. If $h(t) \in V_1$, then $h(t) \in V_1 \cap H_f \subset K \cap H_f$. This implies $h(t) \in \mathrm{Int}\,(K \cap H_f)$, which is a contradiction. So, $h(t) \in \mathrm{Fr}\,K \setminus V_1 \subset \mathrm{Fr}\,K \setminus V = C$, as required. ■

A stronger result holds if (K, L, w) is generated by a finite game. In this case, the duality between weak dominance and the internal-best reply property holds for every $x_0 \in K$. We now introduce some notation and facts that will be used in the proof.

Consider a set C_i with $|C_i| = n+1$ for $n \in \mathcal{N}$. Let $J = \{1, \ldots, n+1\}$ and $X = \Delta(C_i) = \mathrm{co}\,\{e^j \mid j \in J\}$, where e^j is the j-th unit vector of the canonical basis of $\Re^{n+1}$. Let $\alpha \in X$, $\alpha \gg 0$, $K = X - \{\alpha\}$, and $E = \{e^j - \alpha \mid j \in J\}$. So, $K = \mathrm{co}\,E$. Let $\mathcal{K}$ be the n-dimensional linear span of K. As $\alpha \gg 0$, $0 \in \mathrm{Int}\,K$ relative to $\mathcal{K}$.

For $j \in J$, $H_j = \{\sum_{i \in J} p_i(e^i - \alpha) \mid p_j = 0 \wedge \sum_{i \in J} p_i = 1\}$ is the $(n-1)$-dimensional affine span of $E \setminus \{e^j - \alpha\}$, i.e., H_j is a supporting hyperplane of K. Define the linear functional $f_j : \mathcal{K} \to \Re$ by $f_j(.) = -\langle e^j, . \rangle$, where $\langle ., . \rangle$ is the Euclidean inner product on $\mathcal{K}$. The following claims can be verified:

1. $H_j = \{x \in \mathcal{K} \mid f_j(x) = \alpha_j\}$,[10]

2. $\alpha_j = \max f_j(K)$,[11]

3. $\mathrm{Fr}\,K = \cup_{j \in J}(K \cap H_j)$,[12] and

4. $K = \cap_{j \in J} H_j^-$, where $H_j^- = \{x \in \mathcal{K} \mid f_j(x) \leq \alpha_j\}$.[13]

Given these facts, we prove the following result.

Theorem 5.22 *Consider $\Gamma = \{N, (C_j, v_j)_{j \in N}\}$ and $i \in N$ such that N and v_i satisfy Definitions 5.5(a) and (c) respectively, and C_j is nonempty and finite for every $j \in N$. If (K, L, w) is given by Definition 5.15, $x_0 \in K$, and $w(x_0, .) \neq w(x, .)$ for every $x \in K \setminus \{x_0\}$, then x_0 is weakly dominated if and only if it is not an internal-best reply.*

[10] $E \setminus \{e^j - \alpha\}$ contains n linearly independent vectors that are contained in the $(n-1)$-dimensional affine space $\{x \in \mathcal{K} \mid f_j(x) = \alpha_j\}$.

[11] If $x \in K$, then $x = \sum_{i \in J} p_i(e^i - \alpha)$, where each $p_i \geq 0$ and $\sum_{i \in J} p_i = 1$. Then, $f_j(x) = \alpha_j - p_j \leq \alpha_j$ and $\alpha_j = f_j(e^i - \alpha) \in f_j(K)$ for $i \neq j$. So, $\alpha_j = \max f_j(K)$.

[12] Clearly, $\mathrm{Fr}\,K \supset \cup_{j \in J}(K \cap H_j)$. Conversely, let $x \in \mathrm{Fr}\,K \subset K$. Then, there exists $p \in \Re^J_+$ such that $\sum_{i \in J} p_i = 1$ and $x = \sum_{i \in J} p_i(e^i - \alpha)$. If $p_i > 0$ for every $i \in J$, then $x \in \mathrm{Int}\,K$, which contradicts $x \in \mathrm{Fr}\,K$. So, $p_j = 0$ for some $j \in J$. Therefore, $x \in K \cap H_j$.

[13] If $x \in K$, then $f_j(x) \leq \max f_j(K) = \alpha_j$ for every j. So, $K \subset \cap_{j \in J} H_j^-$. Conversely, suppose $x \in \mathcal{K} \setminus K$. As $0 \in \mathrm{Int}\,K$, there exists $t \in (0, 1)$ such that $tx \in \mathrm{Fr}\,K = \cup_{j \in J}(K \cap H_j)$, i.e., $tx \in K \cap H_k$ for some k. So, $t f_k(x) = f_k(tx) = \alpha_k$. Therefore, $f_k(x) = \alpha_k / t > \alpha_k$. It follows that $x \notin H_k^-$, and therefore, $x \notin \cap_{j \in J} H_j^-$.

Proof. As each C_j satisfies Definition 5.5(b), Theorem 5.18 immediately yields the result if $x_0 \in \operatorname{Int} K$. Henceforth, consider $x_0 \in \operatorname{Fr} K$.

The hypotheses of this result, combined with the proof of Theorem 5.18, ensure that (K, L, w) satisfies all the hypotheses of Theorem 2.3. So, the application of Theorem 2.3(B) now only requires verification of its supplementary hypothesis.

1. As $x_0 \in \operatorname{Fr} K$, claim 3 implies $x_0 \in H_j$ for some $j \in J$. Therefore, $n+1 > |J_0| > 0$, where $J_0 = \{j \in J \mid x_0 \in H_j\}$.[14] As each facet $K \cap H_j$ is compact and $|J \setminus J_0| \in \mathcal{N}$, it follows that $C = \cup_{j \in J \setminus J_0}(K \cap H_j)$ is a nonempty and compact subset of $K \setminus \{x_0\}$.

Consider $x \in K \setminus \{x_0\}$. Given x, define $h : [1, \infty) \to \mathcal{K}$ by $h(t) = x_0 + t(x - x_0)$ for $t \geq 1$.

2. Clearly, $1 \in h^{-1}(K)$. As K is compact, it is closed in $\mathcal{K}$. Since h is continuous, $h^{-1}(K)$ is closed in $[1, \infty)$. Suppose $h^{-1}(K)$ is unbounded. Then, there is an unbounded, increasing sequence $(t_n) \subset h^{-1}(K)$ such that $h(t_n) \in K$, i.e., $t_n(x - x_0) \in K - \{x_0\}$, for every $n \in \mathcal{N}$. As the norm is continuous, $\{\|y\| \mid y \in K - \{x_0\}\}$ is compact and therefore bounded. However, $t_n\|x - x_0\| = \|t_n(x - x_0)\| \in \{\|y\| \mid y \in K - \{x_0\}\}$ for every $n \in \mathcal{N}$. As $x \neq x_0$, $\|x - x_0\| > 0$ and $(t_n\|x - x_0\|)$ is unbounded, which is a contradiction. So, $h^{-1}(K)$ is bounded.

Consequently, $h^{-1}(K)$ is nonempty and compact. It follows that there exists $t_0 = \max h^{-1}(K)$, i.e., $t_0 \geq 1$ such that $h(t_0) \in K$ and $h(t) \notin K$ for $t > t_0$. Clearly, $h(t_0) \in \operatorname{Fr} K$. We show that $h(t) \in C$ for some $t \in [1, \infty)$.

3. Suppose $x \in \operatorname{Int} K$ relative to $\mathcal{K}$. Suppose $h(t_0) \notin C$. We derive a contradiction. Since $h(t_0) \in \operatorname{Fr} K = \cup_{j \in J}(K \cap H_j)$ and $h(t_0) \notin C = \cup_{j \in J \setminus J_0}(K \cap H_j)$, we have $t_0 = \max h^{-1}(K) > 1$ and $h(t_0) \in \cup_{j \in J_0}(K \cap H_j)$, i.e., $h(t_0) \in K \cap H_{j_0}$ for some $j_0 \in J_0$. Since $x_0 \in K \cap H_{j_0}$ and $K \cap H_{j_0}$ is convex, $x = t_0^{-1}h(t_0) + (1 - t_0^{-1})x_0 \in K \cap H_{j_0} \subset \operatorname{Fr} K$, which contradicts $x \in \operatorname{Int} K$. So, $h(t_0) \in C$.

So, suppose $x \in \operatorname{Fr} K = \cup_{j \in J}(K \cap H_j)$ relative to $\mathcal{K}$.

4. If $x \in \cup_{j \in J \setminus J_0}(K \cap H_j)$, then $h(1) = x \in C$.

5. Finally, consider $x \in \cup_{j \in J_0}(K \cap H_j)$. Suppose $h(t_0) \notin C$. We derive a contradiction.

   Let $J_x = \{j \in J_0 \mid x \in H_j\}$. For every $j \in J_x$, as $x, x_0 \in H_j$ and H_j is an affine space, $h([1, \infty)) \subset H_j$. It follows that $h([1, \infty)) \subset \cap_{j \in J_x} H_j$.

[14]If $J_0 = J$, then $-\langle e^j, x_0 \rangle = f_j(x_0) = \alpha_j$ for every $j \in J$, i.e., $-\alpha = x_0 \in K$, which implies $0 = x_0 + \alpha \in X$, which is a contradiction. So, $n + 1 > |J_0|$.

(a) Consider $j \in J \setminus J_0$. As $h(t_0) \notin C$, we have $h(t_0) \notin K \cap H_j$. As $h(t_0) \in \mathrm{Fr}\, K \subset K$, we have $h(t_0) \notin H_j$, i.e., $f_j(h(t_0)) < \alpha_j$.

(b) Consider $j \in J_0 \setminus J_x$. As $j \in J_0$, we have $x_0 \in H_j$. As $j \notin J_x$, we have $x \notin H_j$. Therefore, $f_j(x_0) = \alpha_j > f_j(x)$ and $f_j(h(t_0)) = f_j(x_0) + t_0[f_j(x) - f_j(x_0)] < f_j(x_0) = \alpha_j$.

(c) Consider $j \in J_x$. Then, $x_0, x \in H_j$ and so $f_j(x_0) = f_j(x) = \alpha_j = f_j(h(t))$ for every $t \geq 1$.

(d) By steps (a), (b), and the finiteness of $J \setminus J_x$, there exists $\epsilon > 0$ such that $f_j(h(t_0 + \epsilon)) < \alpha_j$ for every $j \in J \setminus J_x$. By step (c), $f_j(h(t_0 + \epsilon)) = \alpha_j$ for every $j \in J_x$. It follows that $h(t_0 + \epsilon) \in \cap_{j \in J} H_j^- = K$, which contradicts the definition of t_0. ■

This proof exploits the fact that, if (K, L, w) is generated by a finite game, then $\mathrm{Fr}\, K$ is the union of a finite number of compact facets. So, for $x_0 \in \mathrm{Fr}\, K$, the set of facets that do not contain x_0 is nonempty and finite. The union of such facets serves as the set C hypothesised in Theorem 2.3(B). But, since these properties do not obtain when the underlying game is not finite, as in Theorem 5.21, a different definition of C is used for $x_0 \in \mathrm{Fr}\, K$ that satisfy a relative interiority condition.

6 Welfare theoretic applications

This section concerns applications of Dualities I and II to decision problems that are not game theoretic in nature and do not involve probabilistic notions such as a belief about the states or a randomisation over actions.

Consider a planner's decision problem $(O, T \times N, v)$ wherein O is an outcome space, T is a parameter space, N is the set of agents, and $v : O \times T \times N \to \Re$ is a function such that $v(., ., i)$ is agent i's utility function. The planner has to choose an outcome from O that solves a multi-objective optimisation problem by meeting an optimality criterion in the form of an efficiency condition.

Definition 6.1 *An outcome $x_0 \in O$ is said to be weakly (resp., strongly) Pareto efficient if it is not strongly (resp., weakly) dominated with respect to the problem $(O, T \times N, v)$, i.e., there is no $x \in O$ such that $v(x_0, ., .) < v(x, ., .)$ (resp., $v(x_0, ., .) \leq v(x, ., .)$ and $v(x_0, ., .) \neq v(x, ., .)$).*

Clearly, an optimal outcome is one that cannot be improved upon by another outcome for every parameter-agent pair. Applying Theorem 2.2 by setting $(X, Y, u) = (O, T \times N, v)$, we have the following duality.

Theorem 6.2 *If $(O, T \times N, v)$ satisfies the hypotheses of Theorem 2.2, then $x_0 \in O$ is weakly Pareto efficient if and only if $v(x_0, \tau, i) \geq v(., \tau, i)$ on O for some $(\tau, i) \in T \times N$.*

Using Theorem 2.3 in an analogous way, we have the following duality.

Theorem 6.3 *Suppose $(O, T \times N, v)$ and $x_0 \in O$ satisfy the hypotheses of Theorem 2.3.*

(A) If x_0 is an interior point of O, then x_0 is strongly Pareto efficient if and only if $v(x_0, \tau, i) \geq v(., \tau, i)$ on O for some (τ, i) that is an internal point of $T \times N$.

(B) Suppose x_0 is a frontier point of O and there is a compact set $C \subset O \setminus \{x_0\}$ such that, for every $x \in O \setminus \{x_0\}$, $x_0 + t(x - x_0) \in C$ for some $t \geq 1$. Then, x_0 is strongly Pareto efficient if and only if $v(x_0, \tau, i) \geq v(., \tau, i)$ on O for some (τ, i) that is an internal point of $T \times N$.

We now turn to Utilitarian efficiency. Given the data O, T, N, and v, define $\hat{v} : O \times T \to \Re^N$ by $\hat{v}(x, \tau)(.) = v(x, \tau, .)$; $\hat{v}(x, \tau)$ is the profile of agents' utilities for outcome x and parameter τ. For $x, y \in \Re^N$, let $x \geq^* y$ (resp., $x =^* y$) if $x(.) \geq y(.)$ (resp., $x(.) = y(.)$) on N. Let $F : \Re^N \to \Re$ be an increasing function, i.e., $x \geq^* y$ and $x \neq^* y$ implies $F(x) > F(y)$ for $x, y \in \Re^N$. Let $w : O \times T \to \Re$ be the utilitarian welfare function given by $w = F \circ \hat{v}$, i.e., F aggregates the profile of agents' utilities $\hat{v}(x, \tau)$ to yield aggregate welfare $w(x, \tau)$.

Definition 6.4 *An outcome $x_0 \in O$ is said to be weakly (resp., strongly) Utilitarian efficient if it is not strongly (resp., weakly) dominated with respect to the problem (O, T, w), i.e., there is no $x \in O$ such that $w(x_0, .) < w(x, .)$ (resp., $w(x_0, .) \leq w(x, .)$ and $w(x_0, .) \neq w(x, .)$) on T.*

As F is increasing, an outcome's weak (resp., strong) Utilitarian efficiency implies its weak (resp., strong) Pareto efficiency.

Suppose X, T, and N are topological spaces, vector addition and scalar multiplication on $\Re^N$ are defined pointwise, and $\Re^N$ is given the compact-open topology. It follows that $\Re^N$ is an LCS (Dugundji [6], Theorem XIII.1.3). Given these conditions, the properties of w can be derived from those of v and F as follows. If v is continuous, then so is $\hat{v}$ (Dugundji [6], Theorem XII.3.1). If F is continuous as well, then so is w. Suppose F is affine. If $x \in O$ and $v(x, ., i)$ is affine for every $i \in N$, then $w(x, .)$ is affine. If $\tau \in T$ and $v(., \tau, i)$ is concave (resp., affine) for every $i \in N$, then $w(., \tau)$ is concave (resp., affine).

Applying Theorem 2.2 to (O, T, w), we have the following duality.

Theorem 6.5 *If (O, T, w) satisfies the hypotheses of Theorem 2.2, then $x_0 \in O$ is weakly Utilitarian efficient if and only if $w(x_0, \tau) \geq w(., \tau)$ on O for some $\tau \in T$.*

Applying Theorem 2.3 similarly, we have the following duality.

Theorem 6.6 *Suppose (O, T, w) and $x_0 \in O$ satisfy the hypotheses of Theorem 2.3.*

(A) If x_0 is an interior point of O, then x_0 is strongly Utilitarian efficient if and only if $w(x_0, \tau) \geq w(., \tau)$ on O for some τ that is an internal point of T.

(B) Suppose x_0 is a frontier point of O and there is a compact set $C \subset O \setminus \{x_0\}$ such that, for every $x \in O \setminus \{x_0\}$, $x_0 + t(x - x_0) \in C$ for some $t \geq 1$. Then, x_0 is strongly Utilitarian efficient if and only if $w(x_0, \tau) \geq w(., \tau)$ on O for some τ that is an internal point of T.

7 Concluding remarks

We have demonstrated dualities between actions that are not strongly (resp., weakly) dominated and actions that are best (resp., internal-best) replies in the setting of action and state spaces that are subsets of abstract topological vector spaces. These results are first applied to a player's decision problem in a many-player game when the players' strategy spaces are subsets of abstract topological vector spaces. They are then applied to a player's decision problem in the σ-additive, the absolutely continuous, and the additive mixed extensions of general many-player games. Unlike the dualities concerning the first two mixed extensions, the duality concerning the third mixed extension can be applied to discontinuous games. Finally, the dualities are used to characterise various welfare theoretic notions of efficient outcomes in terms of different kinds of best replies.

A Appendix

Proof of Lemma 3.2 Consider $x \in K$. As K is metrisable, it is sufficient to find a sequence in K^* converging to x.

1. Suppose $0_{\mathcal{K}} \in K^*$. Let f be the support function of K and $x_n = [(n-1)/n]x$ for $n \in \mathcal{N}$. Then, $f(x) \in [0, 1]$. As K is convex, for every $n \in \mathcal{N}$, $x_n \in K$ and $f(x_n) = f([(n-1)/n]x) = [(n-1)/n]f(x) \in [0, 1)$, i.e., $x_n \in K^*$. As $\mathcal{K}$ is a TVS and $\lim_n (n-1)/n = 1$, the sequence $(x_n) \subset K^*$ converges to x.

2. Suppose $0_{\mathcal{K}} \notin K^*$. As $K^* \neq \emptyset$, there exists $x_0 \in K^*$. Let $L = K - \{x_0\}$. As $\mathcal{K}$ is a TVS, L is a convex and metrisable subset of $\mathcal{K}$. Moreover, $y \in K^*$ if and only if $y - x_0 \in L^*$. So, $0_{\mathcal{K}} \in L^*$. By step 1, as $x - x_0 \in L$, there is a sequence $(y_n) \subset L^*$ converging to $x - x_0$. Therefore, as $\mathcal{K}$ is a TVS, the sequence $(y_n + x_0) \subset K^*$ converges to x. ■

Proof of Lemma 3.4 Consider Y *ex hypothesi.*

1. Suppose $0_{\mathcal{Y}} \in Y^*$. Define $\phi_n : \mathcal{Y} \to \mathcal{Y}$ by $\phi_n(y) = [(n-1)/n]y$ for $n \in \mathcal{N}$ and $y \in \mathcal{Y}$. Let $K_n = \phi_n(Y)$. Clearly, $0_{\mathcal{Y}} = \phi_n(0_{\mathcal{Y}}) \in \phi_n(Y) = K_n$ for every $n \in \mathcal{N}$. K_n is convex as Y is convex and ϕ_n is linear. K_n is compact as ϕ_n is continuous and Y is compact. So, K_n is nonempty, convex, and compact for every $n \in \mathcal{N}$.

   Consider $z \in K_n$ for $n \in \mathcal{N}$. Then, $z = [(n-1)/n]y$ for some $y \in Y$. It follows that $z = [n/(n+1)]y'$ where $y' = (1 - 1/n^2)y$. As Y is convex and $0_{\mathcal{Y}} \in Y$, we have $y' \in Y$. Therefore, $z = \phi_{n+1}(y') \in \phi_{n+1}(Y) = K_{n+1}$. So, $K_n \subset K_{n+1}$ for every $n \in \mathcal{N}$.

   Let f be the support function of Y. Consider $z \in K_n$ for $n \in \mathcal{N}$. Then, $z = \phi_n(y) = [(n-1)/n]y$ for some $y \in Y$. As $y \in Y$, $f(y) \in [0, 1]$. Therefore, $f(z) = f([(n-1)/n]y) = [(n-1)/n]f(y) < 1$. So, $z \in Y^*$. Thus, $K_n \subset Y^*$ for every $n \in \mathcal{N}$. Consequently, $\cup_{n\in\mathcal{N}} K_n \subset Y^*$.

   Conversely, let $z \in Y^*$. So, there exists $\epsilon > 0$ such that $|\delta| < \epsilon$ implies $(1+\delta)z = z+\delta z \in Y$. Let $n \in \mathcal{N}$ be such that $n > 1$ and $1/(n-1) < \epsilon$. Then, $y = [1 + 1/(n-1)]z \in Y$ and $z = [(n-1)/n]y = \phi_n(y) \in K_n$. Thus, $Y^* \subset \cup_{n\in\mathcal{N}} K_n$.

   So, $\{K_n \mid n \in \mathcal{N}\}$ internally approximates Y.

2. Suppose $0_{\mathcal{Y}} \notin Y^*$. As $Y^* \neq \emptyset$, there exists $y_0 \in Y^*$. Define $T : \mathcal{Y} \to \mathcal{Y}$ by $T(y) = y - y_0$. As $\mathcal{Y}$ is a TVS, T is an affine homeomorphism and $0_{\mathcal{Y}} \in T(Y)^*$. So, $T(Y)$ is a convex and compact subset of $\mathcal{Y}$. By step 1, there is a family of sets $\{K_n \mid n \in \mathcal{N}\}$ that internally approximates $T(Y)$. The function inverse of T is T^{-1}, given by $T^{-1}(y) = y + y_0$. We verify that the family of sets $\{T^{-1}(K_n) \mid n \in \mathcal{N}\}$ internally approximates Y.

   Consider $n \in \mathcal{N}$. $T^{-1}(K_n) \neq \emptyset$ as $K_n \neq \emptyset$. Since K_n is convex and T^{-1} is affine, $T^{-1}(K_n)$ is convex. As K_n is compact and T^{-1} is continuous, $T^{-1}(K_n)$ is compact. As $K_n \subset K_{n+1}$, we have $T^{-1}(K_n) \subset T^{-1}(K_{n+1})$. Finally, as $T(Y)^* = \cup_{n\in\mathcal{N}} K_n$, we have $T^{-1}(T(Y)^*) = \cup_{n\in\mathcal{N}} T^{-1}(K_n)$. So, it suffices to show that $T^{-1}(T(Y)^*) = Y^*$, i.e., $T(Y)^* = T(Y^*)$.

   Consider $y \in T(Y^*)$ and $z \in \mathcal{Y}$. Then, $y = x - y_0$ for some $x \in Y^* \subset Y$. So, $y \in T(Y)$. As $x \in Y^*$, there exists $\epsilon > 0$ such that $x + \delta z \in Y$ for $|\delta| < \epsilon$, which implies $y + \delta z = x + \delta z - y_0 \in T(Y)$ for $|\delta| < \epsilon$. Thus, $y \in T(Y)^*$. So, $T(Y^*) \subset T(Y)^*$.

   Conversely, let $y \in T(Y)^* \subset T(Y)$. Then, $y = T(x) = x - y_0$ for some $x \in Y$. Consider $z \in \mathcal{Y}$. As $y \in T(Y)^*$, there exists $\epsilon > 0$ such that $y + \delta z \in T(Y)$ for $|\delta| < \epsilon$, which implies $x + \delta z = y + \delta z + y_0 \in Y$ for $|\delta| < \epsilon$. Thus, $x \in Y^*$ and $y = T(x) \in T(Y^*)$. So, $T(Y)^* \subset T(Y^*)$. ■

Proof of Lemma 3.5 Consider $\mathcal{E}$ and $\mathcal{A}$ *ex hypothesi.*

1. Since $\emptyset \in \mathcal{E}$ and $\mathcal{E} \subset \mathcal{A}$, we have $\emptyset \in \mathcal{A}$.

2. Consider $E \in \mathcal{E}$. Since $E \subset T$ and $T \in \mathcal{E}$, property (c) of a semialgebra implies that there are sets $E_0, \ldots, E_n \in \mathcal{E}$ such that $E = E_0 \subset E_1 \subset \ldots \subset E_n = T$ and $E_i \setminus E_{i-1} \in \mathcal{E}$ for $i = 1, \ldots, n$. As $E_i \setminus E_{i-1}$, $i = 1, \ldots, n$, are pairwise disjoint and $T \setminus E = \cup_{i=1}^n (E_i \setminus E_{i-1})$, $T \setminus E \in \mathcal{A}$.

3. Let $E, F \in \mathcal{A}$. Then, there are pairwise disjoint sets $E_1, \ldots, E_n \in \mathcal{E}$ and pairwise disjoint sets $F_1, \ldots, F_m \in \mathcal{E}$ such that $E = \cup_{i=1}^n E_i$ and $F = \cup_{j=1}^m F_j$. Hence, $E \cap F = \cup_{i=1}^n \cup_{j=1}^m (E_i \cap F_j)$. As the sets $E_i \cap F_j \in \mathcal{E}$ are pairwise disjoint for $(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$, we have $E \cap F \in \mathcal{A}$.

4. Consider $E \in \mathcal{A}$. Then, there are pairwise disjoint sets $E_1, \ldots, E_n \in \mathcal{E}$ such that $E = \cup_{i=1}^n E_i$. Hence, $T \setminus E = \cap_{i=1}^n (T \setminus E_i)$. By step 2, $T \setminus E_i \in \mathcal{A}$ for $i = 1, \ldots, n$. Using step 3, $T \setminus E \in \mathcal{A}$.

5. Consider $E, F \in \mathcal{A}$. Then, $F \setminus E = F \cap (T \setminus E)$. By step 4, $T \setminus E \in \mathcal{A}$. As $F \in \mathcal{A}$, step 3 implies, $F \setminus E \in \mathcal{A}$.

6. Consider $E, F \in \mathcal{A}$. Then, $E \cup F = E \cup (F \setminus E)$, where $E, F \setminus E \in \mathcal{A}$ by step 5. As E and $F \setminus E$ are disjoint, there are pairwise disjoint sets $E_1, \ldots, E_m, E_{m+1}, \ldots, E_n \in \mathcal{E}$ such that $E = \cup_{i=1}^m E_i$ and $F \setminus E = \cup_{i=m+1}^n E_i$. Hence, $E \cup F = E \cup (F \setminus E) = \cup_{i=1}^n E_i \in \mathcal{A}$.

Combining steps 1, 4, and 6, $\mathcal{A}$ is an algebra on T. Consider $E \in \mathcal{A}$. Then, there are sets $E_1, \ldots, E_n \in \mathcal{E}$ such that $E = \cup_{i=1}^n E_i$. If $\mathcal{T}$ is an algebra on T and $\mathcal{E} \subset \mathcal{T}$, then $E_1, \ldots, E_n \in \mathcal{T}$ and $E = \cup_{i=1}^n E_i \in \mathcal{T}$. Thus, $\mathcal{A} \subset \mathcal{T}$. ■

Proof of Lemma 5.7 Consider $(\Omega, \mathcal{F}, P)$, L^p_Ω, and D *ex hypothesi.*

1. $(L^p_\Omega, \|.\|_{p,\Omega})$ is a reflexive Banach space (Dunford and Schwartz [7], Theorem III.6.6 and Corollary IV.8.2); it is separable by assumption.

Henceforth, consider L^p_Ω with the weak topology. Let $q = p/(p-1) > 1$.

2. Then, L^p_Ω is a Hausdorff LCS (Dunford and Schwartz [7], Lemma V.3.3) and B^p_Ω is compact (Dunford and Schwartz [7], Theorem V.4.7), metrisable (Dunford and Schwartz [7], Theorem V.6.3), and closed.

3. D is nonempty as $1_\Omega \in D$. Convexity of D is routinely verified. D inherits its metrisability from B^p_Ω.

4. Let x be an accumulation point of D. Then, there is a sequence $(x_n) \subset D$ converging to x. Since $(x_n) \subset B^p_\Omega$ and B^p_Ω is closed, $x \in B^p_\Omega$. Consider $E \in \mathcal{F}$. The mapping $L^p_\Omega \ni x \mapsto \int_E P(d\omega)\, x(\omega)$ is a continuous real-valued linear functional as $1_E \in L^q_\Omega$ (Dunford and Schwartz [7], Theorem IV.8.1). Therefore, $\int_E P(d\omega)\, x(\omega) = \lim_n \int_E P(d\omega)\, x_n(\omega)$.

As $(x_n) \subset D$, $\int_E P(d\omega)\, x_n(\omega) \geq 0$ and $\int_\Omega P(d\omega)\, x_n(\omega) = 1$ for every $n \in \mathcal{N}$. Hence, $\int_E P(d\omega)\, x(\omega) \geq 0$ and $\int_\Omega P(d\omega)\, x(\omega) = 1$. Thus, $x \in D$ and D is a closed subset of B^p_Ω. As B^p_Ω is compact, so is D. ■

Proof of Lemma 5.8 Consider w, φ, and Φ *ex hypothesi*. Let $q = p/(p-1) > 1$. As w is bounded, $|w(.,.)| < \beta$ on $\Omega \times \Theta$ for some $\beta > 0$.

(A) Consider $\omega \in \Omega$. As $w(\omega, .)$ is bounded and measurable, $w(\omega, .) \in L^q_\Theta$. Then, $L^p_\Theta \ni h \mapsto \int_\Theta Q(d\theta)\, w(\omega, \theta)h(\theta)$ is a continuous real-valued linear functional (Dunford and Schwartz [7], Theorem IV.8.1). The restriction of this functional to Y coincides with $\varphi(\omega, .)$. Therefore, $|\varphi(\omega, y)| \leq \beta$ for $y \in Y$ and $\varphi(\omega, .)$ is continuous. Also, $\varphi(., y)$ is measurable for $y \in Y$ (Rao [11], Theorem 6.2.1, page 327).

(B) Consider $(x, y) \in X \times Y$. Since x, y, and the projections $\pi_1 : \Omega \times \Theta \to \Omega$ and $\pi_2 : \Omega \times \Theta \to \Theta$, are measurable, so are $x \circ \pi_1$ and $y \circ \pi_2$. As products of measurable functions are measurable, so is $xy : \Omega \times \Theta \to \Re$, given by $xy(\omega, \theta) = x(\omega)y(\theta)$. By Tonelli's theorem (Rao [11], Theorem 6.2.2), $\int_{\Omega\times\Theta} P \times Q(d\omega, d\theta)\, |xy(\omega, \theta)|^p = \int_\Omega P(d\omega)\, |x(\omega)|^p \int_\Theta Q(d\theta)\, |y(\theta)|^p < \infty$. So, $xy \in L^p_{\Omega\times\Theta}$.

  (i) As w is bounded and measurable, $w \in L^q_{\Omega\times\Theta}$. Therefore, $\int_{\Omega\times\Theta} P \times Q(d\omega, d\theta)\, w(\omega, \theta)h(\omega, \theta) \in \Re$ for every $h \in L^p_{\Omega\times\Theta}$ (Dunford and Schwartz [7], Theorem IV.8.1). Setting $h = xy \in L^p_{\Omega\times\Theta}$, we have $\Phi(x, y) \in \Re$ and $|\Phi(x, y)| \leq \beta$ for $(x, y) \in X \times Y$.

  (ii) By the Fubini-Stone theorem (Rao [11], Theorem 6.2.1), $\Phi(x, y) = \int_\Omega P(d\omega)\, x(\omega) \int_\Theta Q(d\theta)\, w(\omega, \theta)y(\theta) = \int_\Omega P(d\omega)\, x(\omega)\varphi(\omega, y)$.

  (iii) Since the weak topology of Y is metrisable by Lemma 5.7, if $\Phi(x, y) = \lim_n \Phi(x, y_n)$ for every sequence $(y_n) \subset Y$ converging to $y \in Y$, then $\Phi(x, .)$ is continuous at y.

  Consider a sequence $(y_n) \subset Y$ converging to $y \in Y$. By (A), we have $\varphi(\omega, y_n), \varphi(\omega, y) \in \Re$ and $\lim_n \varphi(\omega, y_n) = \varphi(\omega, y)$ for every $\omega \in \Omega$, and the functions $\varphi(., y)$ and $\varphi(., y_n)$ are measurable for every n. As x is measurable, $\beta x(.)$, $x(.)\varphi(., y)$, and $x(.)\varphi(., y_n)$ are measurable for every n. Since $\beta x(.)$ is P-integrable, and $|x(.)\varphi(., y_n)| \leq \beta x(.)$ P-a.e. for every n, and $\lim_n x(.)\varphi(., y_n) = x(.)\varphi(., y)$ pointwise, we have $\Phi(x, y) = \int_\Omega P(d\omega)\, x(\omega)\varphi(\omega, y) = \lim_n \int_\Omega P(d\omega)\, x(\omega)\varphi(\omega, y_n) = \lim_n \Phi(x, y_n)$ by (ii) and the dominated convergence theorem (Rao [11], Theorem 4.3.5). So, $\Phi(x, .)$ is continuous at y. As this holds for every $y \in Y$, $\Phi(x, .)$ is continuous on Y.

  (iv) Analogously, $\Phi(., y)$ is continuous on X.

(C) Consider $y \in Y$ and $\epsilon > 0$. By Lemma 5.7, the weak topology of Y is metrised by some metric d. Since the functions in $\{\varphi(\omega, .) \mid \omega \in \Omega\}$

are equicontinuous at y, there exists $\delta > 0$ such that, if $y' \in Y$ and $d(y', y) < \delta$, then $|\varphi(\omega, y') - \varphi(\omega, y)| < \epsilon$ for every $\omega \in \Omega$.

(i) Consider a sequence $(y_n) \subset Y$ converging to $y \in Y$. Then, there exists $N \in \mathcal{N}$ such that $n \geq N$ implies $d(y_n, y) < \delta$, and therefore, $|\varphi(\omega, y_n) - \varphi(\omega, y)| < \epsilon$ for every $\omega \in \Omega$. So, for every $x \in X$, $n \geq N$ implies $|\Phi(x, y_n) - \Phi(x, y)| = |\int_\Omega P(d\omega)\, x(\omega)\varphi(\omega, y_n) - \int_\Omega P(d\omega)\, x(\omega)\varphi(\omega, y)| \leq \int_\Omega P(d\omega)\, x(\omega)|\varphi(\omega, y_n) - \varphi(\omega, y)| \leq \epsilon$. Thus, $(\Phi(x, y_n))$ converges to $\Phi(x, y)$ uniformly in $x \in X$.

As X and Y are metrisable by Lemma 5.7, so is $X \times Y$. Therefore, if $\lim_n \Phi(x_n, y_n) = \Phi(x_0, y_0)$ for every $(x_0, y_0) \in X \times Y$ and every sequence $(x, y) : \mathcal{N} \to X \times Y$ converging to (x_0, y_0), then Φ is continuous.

Consider a sequence $(x, y) : \mathcal{N} \to X \times Y$ converging to $(x_0, y_0) \subset X \times Y$. For $(i_1, j_1), (i_2, j_2) \in \mathcal{N} \times \mathcal{N}$, let $(i_1, j_1) \geq^* (i_2, j_2)$ if $i_1 \geq i_2$ and $j_1 \geq j_2$. Then, $\geq^*$ is a latticial ordering on $\mathcal{N} \times \mathcal{N}$ and $(\mathcal{N} \times \mathcal{N}, \geq^*)$ is a directed set. Define the net $z : \mathcal{N} \times \mathcal{N} \to X \times Y$ by $z(i, j) = (x_i, y_j)$. Since (x_n) converges to x_0 and (y_n) converges to y_0, the net z converges to (x_0, y_0). We show that $\lim_{(i,j)} \Phi(x_i, y_j) = \Phi(x_0, y_0)$, and so, $\lim_n \Phi(x_n, y_n) = \Phi(x_0, y_0)$, as required. Consider $\epsilon > 0$.

(ii) By (i), there exists $N \in \mathcal{N}$ such that $j \geq N$ implies $|\Phi(x_i, y_j) - \Phi(x_i, y_0)| < \epsilon/8$ for every x_i, $i \in \mathcal{N}$. So, $j \geq N$ implies $|\Phi(x_i, y_j) - \Phi(x_i, y_N)| \leq |\Phi(x_i, y_j) - \Phi(x_i, y_0)| + |\Phi(x_i, y_0) - \Phi(x_i, y_N)| < \epsilon/4$ for every x_i, $i \in \mathcal{N}$.

(iii) By (B), there exists $M \in \mathcal{N}$ such that $i \geq M$ implies $|\Phi(x_0, y_N) - \Phi(x_i, y_N)| < \epsilon/8$. Hence, $i \geq M$ implies $|\Phi(x_i, y_N) - \Phi(x_M, y_N)| \leq |\Phi(x_i, y_N) - \Phi(x_0, y_N)| + |\Phi(x_0, y_N) - \Phi(x_M, y_N)| < \epsilon/4$.

(iv) If $(i, j) \geq^* (M, N)$, then by (ii) and (iii), $|\Phi(x_i, y_j) - \Phi(x_M, y_N)| \leq |\Phi(x_i, y_j) - \Phi(x_i, y_N)| + |\Phi(x_i, y_N) - \Phi(x_M, y_N)| < \epsilon/2$.

(v) By (iv), if $(i_1, j_1) \geq^* (M, N)$ and $(i_2, j_2) \geq^* (M, N)$, then we have $|\Phi(x_{i_1}, y_{j_1}) - \Phi(x_{i_2}, y_{j_2})| \leq |\Phi(x_{i_1}, y_{j_1}) - \Phi(x_M, y_N)| + |\Phi(x_M, y_N) - \Phi(x_{i_2}, y_{j_2})| < \epsilon$.

(vi) By (v), $(\Phi(x_i, y_j))$ is a Cauchy net in $\Re$. Since $\Re$ is complete, $\lim_{(i,j)} \Phi(x_i, y_j) = r$ for some $r \in \Re$ (Dugundji [6], Theorem XIV.3.2).

(vii) Therefore, $|r - \Phi(x_{i_2}, y_{j_2})| = |\lim_{(i_1,j_1)} \Phi(x_{i_1}, y_{j_1}) - \Phi(x_{i_2}, y_{j_2})| = \lim_{(i_1,j_1)} |\Phi(x_{i_1}, y_{j_1}) - \Phi(x_{i_2}, y_{j_2})| \leq \epsilon$ for every $(i_2, j_2) \geq^* (M, N)$, by applying (v) and (vi).

(viii) Consider $\eta > 0$. By (i), there exists $N' \geq N$ such that $j_2 \geq N'$ implies $|\Phi(x_{i_2}, y_{j_2}) - \Phi(x_{i_2}, y_0)| \leq \eta$ for $i_2 \geq M$.

Hence, by (vii), $|r - \Phi(x_{i_2}, y_0)| \leq |r - \Phi(x_{i_2}, y_{j_2})| + |\Phi(x_{i_2}, y_{j_2}) - \Phi(x_{i_2}, y_0)| \leq \epsilon + \eta$ for $(i_2, j_2) \geq^* (M, N')$. Letting $\eta \downarrow 0$, we have $|r - \Phi(x_{i_2}, y_0)| \leq \epsilon$ for $i_2 \geq M$.

(ix) Hence, (B) and (viii) imply $|r - \Phi(x_0, y_0)| = |r - \lim_{i_2} \Phi(x_{i_2}, y_0)| = \lim_{i_2} |r - \Phi(x_{i_2}, y_0)| \leq \epsilon$. Letting $\epsilon \downarrow 0$, we have $r = \Phi(x_0, y_0)$.

(D) Consider $x \in X$, $y, y' \in Y$, and $t \in (0, 1)$. As Y is convex by Lemma 5.7, $ty + (1-t)y' \in Y$. Evidently, $\varphi(\omega, ty + (1-t)y') = t\varphi(\omega, y) + (1-t)\varphi(\omega, y')$ for every $\omega \in \Omega$. Multiplying by $x(\omega)$ and integrating with respect to P, it follows that $\Phi(x, ty + (1-t)y') = t\Phi(x, y) + (1-t)\Phi(x, y')$. The other part is proved analogously. ■

Proof of Lemma 5.12 Consider Γ, $i \in N$, and $\{\mathcal{C}_j \mid j \in N\}$ *ex hypothesi.*

(A) Consider a net $(x_j^\alpha)_{j\in N} \subset \prod_{j\in N} \Delta(\mathcal{C}_j)$ that converges to $(x_j)_{j\in N} \in \prod_{j\in N} \Delta(\mathcal{C}_j)$. Let $\bar{x}^\alpha$ and $\bar{x}$ be the product measures generated by $(x_j^\alpha)_{j\in N}$ and $(x_j)_{j\in N}$ respectively. We show that $(\bar{x}^\alpha)$ converges to $\bar{x}$.

(i) Consider $j \in N$. Then, (x_j^α) converges to x_j. For every $E \in \mathcal{C}_j$, since $1_E \in B(C_j, \mathcal{C}_j)$, $\lim_\alpha x_j^\alpha(E) = \lim_\alpha \int_{C_j} x_j^\alpha(dc_j)\, 1_E(c_j) = \int_{C_j} x_j(dc_j)\, 1_E(c_j) = x_j(E)$.

(ii) Let $E \in \mathcal{E}$. Then, $E = \prod_{j\in N} E_j$ where $E_j \in \mathcal{C}_j$ for every $j \in N$. Hence, $\bar{x}^\alpha(E) = \prod_{j\in N} x_j^\alpha(E_j)$ and $\bar{x}(E) = \prod_{j\in N} x_j(E_j)$. By step (i), $\lim_\alpha \bar{x}^\alpha(E) = \prod_{j\in N} \lim_\alpha x_j^\alpha(E_j) = \prod_{j\in N} x_j(E_j) = \bar{x}(E)$.

(iii) Let $E \in \mathcal{C}$. Then, there are pairwise disjoint sets $E_1, \ldots, E_n \in \mathcal{E}$ such that $E = \cup_{j=1}^n E_j$. So, $\bar{x}^\alpha(E) = \sum_{j=1}^n \bar{x}^\alpha(E_j)$ and $\bar{x}(E) = \sum_{j=1}^n \bar{x}(E_j)$. By step (ii), $\lim_\alpha \bar{x}^\alpha(E) = \sum_{j=1}^n \lim_\alpha \bar{x}^\alpha(E_j) = \sum_{j=1}^n \bar{x}(E_j) = \bar{x}(E)$.

(iv) Consider $f = \sum_{k=1}^n a_k 1_{E_k}$ with $a_1, \ldots, a_n \in \Re$ and $E_1, \ldots, E_n \in \mathcal{C}$. The integral's linearity (Dunford and Schwartz [7], Theorem III.2.19) implies that $\int_C \bar{x}^\alpha(dc)\, f(c) = \sum_{k=1}^n a_k \bar{x}^\alpha(E_k)$ and $\int_C \bar{x}(dc)\, f(c) = \sum_{k=1}^n a_k \bar{x}(E_k)$. Using step (iii), $\int_C \bar{x}(dc)\, f(c) = \sum_{k=1}^n a_k \bar{x}(E_k) = \sum_{k=1}^n a_k \lim_\alpha \bar{x}^\alpha(E_k) = \lim_\alpha \int_C \bar{x}^\alpha(dc)\, f(c)$.

(v) Consider $f \in B(C, \mathcal{C})$. Then, there is a sequence of step functions (f_n) converging uniformly to f. Fix $\epsilon > 0$. Then, there exists $n_0 \in \mathcal{N}$ such that $n \geq n_0$ implies $\sup\{|f_n(c) - f(c)| \mid c \in C\} \leq \epsilon$, i.e., $|f_n(c) - f(c)| \leq \epsilon$ for every $c \in C$.

Observe that $\int_C \bar{x}^\alpha(dc)\, f(c) - \int_C \bar{x}(dc)\, f(c) = \int_C \bar{x}^\alpha(dc)\, [f(c) - f_n(c)] + \int_C \bar{x}^\alpha(dc)\, f_n(c) - \int_C \bar{x}(dc)\, [f(c) - f_n(c)] - \int_C \bar{x}(dc)\, f_n(c)$. Hence, $|\int_C \bar{x}^\alpha(dc)\, f(c) - \int_C \bar{x}(dc)\, f(c)| \leq \int_C \bar{x}^\alpha(dc)\, |f(c) - f_n(c)| + \left|\int_C \bar{x}^\alpha(dc)\, f_n(c) - \int_C \bar{x}(dc)\, f_n(c)\right| + \int_C \bar{x}(dc)\, |f(c) - f_n(c)|$.

Consider $n \geq n_0$. Consequently, $\int_C \bar{x}^\alpha(dc)\, |f(c) - f_n(c)| \leq \epsilon$ and $\int_C \bar{x}(dc)\, |f(c) - f_n(c)| \leq \epsilon$. So, $|\int_C \bar{x}^\alpha(dc)\, f(c) - \int_C \bar{x}(dc)\, f(c)| \leq$

$|\int_C \bar{x}^\alpha(dc)\, f_n(c) - \int_C \bar{x}(dc)\, f_n(c)| + 2\epsilon$. Since f_n is a step function, step (iv) implies $\lim_\alpha \int_C \bar{x}^\alpha(dc)\, f_n(c) = \int_C \bar{x}(dc)\, f_n(c)$. Hence, $|\lim_\alpha \int_C \bar{x}^\alpha(dc)\, f(c) - \int_C \bar{x}(dc)\, f(c)| \leq 2\epsilon$. Letting $\epsilon \downarrow 0$ yields $\lim_\alpha \int_C \bar{x}^\alpha(dc)\, f(c) = \int_C \bar{x}(dc)\, f(c)$, i.e., $(\bar{x}^\alpha)$ converges to $\bar{x}$.

(B) Consider $j \in N \setminus \{i\}$. As $w_j \in B(C_i \times C_j, \mathcal{C}_i \times \mathcal{C}_j)$, there is a sequence $(f_n) \subset B(C_i \times C_j, \mathcal{C}_i \times \mathcal{C}_j)$ of step functions converging uniformly to w_j, i.e., given $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $n \geq n_0$ implies $|f_n(c_i, c_j) - w_j(c_i, c_j)| \leq \epsilon$ for every $(c_i, c_j) \in C_i \times C_j$, and consequently, $|f_n \circ \pi_{i,j}(c) - w_j \circ \pi_{i,j}(c)| \leq \epsilon$ for every $c \in C$. Since f_n is a step function on $C_i \times C_j$, $f_n \circ \pi_{i,j}$ is a step function on C. Therefore, the sequence of step functions $(f_n \circ \pi_{i,j}) \subset B(C, \mathcal{C})$ converges to $w_j \circ \pi_{i,j}$ uniformly on C. It follows that $w_j \circ \pi_{i,j} \in B(C, \mathcal{C})$.

(C) Given $z \in \Delta(\mathcal{C})$, the functional $B(C, \mathcal{C}) \ni g \mapsto \int_C z(dc)\, g(c)$ is continuous with respect to the supremum norm topology on $B(C, \mathcal{C})$. Similarly, for every $z \in \Delta(\mathcal{C}_i \times \mathcal{C}_j)$, the linear functional $B(C_i \times C_j, \mathcal{C}_i \times \mathcal{C}_j) \ni g \mapsto \int_{C_i \times C_j} z(dc_i, dc_j)\, g(c_i, c_j)$ is continuous with respect to the supremum norm topology on $B(C_i \times C_j, \mathcal{C}_i \times \mathcal{C}_j)$.

Consider $(x_k)_{k \in N} \in \prod_{k \in N} \Delta(\mathcal{C}_k)$ and the resulting product measure $\bar{x} \in \Delta(\mathcal{C})$.

(i) By (B), $(f_n \circ \pi_{i,j}) \subset B(C, \mathcal{C})$ converges uniformly to $w_j \circ \pi_{i,j} \in B(C, \mathcal{C})$. Since $\bar{x} \in \Delta(\mathcal{C})$, we have $\lim_n \int_C \bar{x}(dc)\, f_n \circ \pi_{i,j}(c) = \int_C \bar{x}(dc)\, w_j \circ \pi_{i,j}(c)$.

(ii) Since $(f_n) \subset B(C_i \times C_j, \mathcal{C}_i \times \mathcal{C}_j)$ converges uniformly to $w_j \in B(C_i \times C_j, \mathcal{C}_i \times \mathcal{C}_j)$ and $\bar{x} \circ \pi_{i,j}^{-1} \in \Delta(\mathcal{C}_i \times \mathcal{C}_j)$, $\lim_n \int_{C_i \times C_j} \bar{x} \circ \pi_{i,j}^{-1}(dc_i, dc_j)\, f_n(c_i, c_j) = \int_{C_i \times C_j} \bar{x} \circ \pi_{i,j}^{-1}(dc_i, dc_j)\, w_j(c_i, c_j)$.

(iii) By elementary calculations, $\int_{C_i \times C_j} \bar{x} \circ \pi_{i,j}^{-1}(dc_i, dc_j)\, f_n(c_i, c_j) = \int_C \bar{x}(dc)\, f_n \circ \pi_{i,j}(c)$ for every $n \in \mathcal{N}$. Combining this with steps (i) and (ii) yields $\int_{C_i \times C_j} \bar{x} \circ \pi_{i,j}^{-1}(dc_i, dc_j)\, w_j(c_i, c_j) = \lim_n \int_{C_i \times C_j} \bar{x} \circ \pi_{i,j}^{-1}(dc_i, dc_j)\, f_n(c_i, c_j) = \lim_n \int_C \bar{x}(dc)\, f_n \circ \pi_{i,j}(c) = \int_C \bar{x}(dc)\, w_j \circ \pi_{i,j}(c)$. ■

Proof of Lemma 5.13 Consider Γ, $i \in N$, and $\{\mathcal{C}_j \mid j \in N\}$ *ex hypothesi.*

(A) By Lemma 5.12, $w_j \circ \pi_{i,j} \in B(C, \mathcal{C})$ for every $j \in N \setminus \{i\}$. As $B(C, \mathcal{C})$ is a linear space, $v_i \in B(C, \mathcal{C})$. Hence, $\Delta(\mathcal{C}) \ni z \mapsto \int_C z(dc)\, v_i(c)$ is continuous. By Lemma 5.12, $\prod_{j \in N} \Delta(\mathcal{C}_j) \ni (x_j)_{j \in N} \mapsto \bar{x} \in \Delta(\mathcal{C})$ is continuous. By composing these mappings, V_i is continuous.

(B) Consider $(x, y) \in \Delta(\mathcal{C}_i) \times \prod_{j \in N \setminus \{i\}} \Delta(\mathcal{C}_j)$ and let $\bar{y} = \prod_{j \in N \setminus \{i\}} y_j$. Then, $V_i(x, y) = \int_C x \times \bar{y}(dc) \sum_{j \in N \setminus \{i\}} w_j \circ \pi_{i,j}(c) = \sum_{j \in N \setminus \{i\}} \int_C x \times$

$\bar{y}(dc)\, w_j \circ \pi_{i,j}(c)$ by the integral's linearity (Dunford and Schwartz [7], Theorem III.2.19). Consider $j \in N \setminus \{i\}$. As $(x \times \bar{y}) \circ \pi_{i,j}^{-1} = x \times y_j$ on $\mathcal{E}_{i,j}$, $(x \times \bar{y}) \circ \pi_{i,j}^{-1} = x \times y_j$ on $\mathcal{C}_i \times \mathcal{C}_j$ (Dunford and Schwartz [7], Lemma III.11.1). By Lemma 5.12, $\int_C x \times \bar{y}(dc)\, w_j \circ \pi_{i,j}(c) = \int_{C_i \times C_j} (x \times \bar{y}) \circ \pi_{i,j}^{-1}(dc_i, dc_j)\, w_j(c_i, c_j) = \int_{C_i \times C_j} x \times y_j(dc_i, dc_j)\, w_j(c_i, c_j)$. Hence, $V_i(x, y) = \sum_{j \in N \setminus \{i\}} \int_{C_i \times C_j} x \times y_j(dc_i, dc_j)\, w_j(c_i, c_j)$.

Consider $x \in \Delta(\mathcal{C}_i)$, $y, z \in \prod_{j \in N \setminus \{i\}} \Delta(\mathcal{C}_j)$, $t \in (0, 1)$, and $j \in N \setminus \{i\}$. Then, $ty_j + (1-t)z_j \in \Delta(\mathcal{C}_j)$. Since $x \times [ty_j + (1-t)z_j]$ and $t(x \times y_j) + (1-t)(x \times z_j)$ agree on $\mathcal{E}_{i,j}$, they agree on $\mathcal{C}_i \times \mathcal{C}_j$. So, $\int_{C_i \times C_j} x \times [ty_j + (1-t)z_j](dc_i, dc_j)\, w_j(c_i, c_j) = \int_{C_i \times C_j} [t(x \times y_j) + (1-t)(x \times z_j)](dc_i, dc_j)\, w_j(c_i, c_j) = t \int_{C_i \times C_j} x \times y_j(dc_i, dc_j)\, w_j(c_i, c_j) + (1-t) \int_{C_i \times C_j} x \times z_j(dc_i, dc_j)\, w_j(c_i, c_j)$. Summing over $j \in N \setminus \{i\}$ implies $V_i(x, ty + (1-t)z) = tV_i(x, y) + (1-t)V_i(x, z)$.

(C) Consider $x, y \in \Delta(\mathcal{C}_i)$, $z \in \prod_{j \in N \setminus \{i\}} \Delta(\mathcal{C}_j)$, $t \in (0, 1)$, and $j \in N \setminus \{i\}$. Then, $tx + (1-t)y \in \Delta(\mathcal{C}_i)$. Since $[tx + (1-t)y] \times z_j$ and $t(x \times z_j) + (1-t)(y \times z_j)$ agree on $\mathcal{E}_{i,j}$, they agree on $\mathcal{C}_i \times \mathcal{C}_j$. So, $\int_{C_i \times C_j} [tx + (1-t)y] \times z_j(dc_i, dc_j)\, w_j(c_i, c_j) = \int_{C_i \times C_j} [t(x \times z_j) + (1-t)(y \times z_j)](dc_i, dc_j)\, w_j(c_i, c_j) = t \int_{C_i \times C_j} x \times z_j(dc_i, dc_j)\, w_j(c_i, c_j) + (1-t) \int_{C_i \times C_j} y \times z_j(dc_i, dc_j)\, w_j(c_i, c_j)$. Summing over $j \in N \setminus \{i\}$ implies $V_i(tx + (1-t)y, z) = tV_i(x, z) + (1-t)V_i(y, z)$. ■

Proof of Lemma 5.17 Let $K = \Delta(T) - \{\alpha\}$. Then, K is convex, $0 \in K$, and $K \subset B_2 = \{x \in \mathrm{rca}(T) \mid \|x\| \leq 2\}$. As $\Delta(T)$ is a compact, metrisable, and separable subset of $\mathrm{rca}(T)$ (Parthasarathy [9], Theorem II.6.4), so is K. Let K' be a countable set that is dense in K.

1. Given its subspace topology derived from $\mathrm{rca}(T)$, $\mathcal{S}$ is a Hausdorff LCS. This establishes (A).

Now we prove (B).

2. Let $\{h_i \mid i \in I\}$ be the family of non-zero continuous real-valued linear functionals on $\mathcal{S}$. For every $i \in I$, since $0 \in K$, h_i generates the closed halfspace $H_i^- = \{x \in \mathcal{S} \mid h_i(x) \leq c_i\} \supset K$ where $c_i = \max h_i(K) \geq h_i(0) = 0$. It follows that $K = \cap_{i \in I} H_i^-$ (Schaefer [12], Theorem II.10.1).

3. As B_2 is metrisable, so is $\mathcal{S} \cap B_2$. Since K is a compact subset of $\mathcal{S} \cap B_2$, K is closed in $\mathcal{S} \cap B_2$. Setting $H_i^+ = (\mathcal{S} \cap B_2) \setminus H_i^-$, we have $\mathrm{Fr}\, K = \overline{K} \cap \overline{(\mathcal{S} \cap B_2) \setminus K} = K \cap \overline{(\mathcal{S} \cap B_2) \setminus \cap_{i \in I} H_i^-} = K \cap \overline{\cup_{i \in I} H_i^+}$.

4. As H_i^+ is open in $\mathcal{S} \cap B_2$, $\mathcal{H} = \{H_i^+ \mid i \in I\}$ is an open cover of $(\mathcal{S} \cap B_2) \setminus K$. As $(\mathcal{S} \cap B_2) \setminus K$ is metrisable, Stone's theorem (Dugundji [6],

Theorem IX.5.3) implies it is paracompact. Therefore, $\mathcal{H}$ has a locally finite refinement consisting of open sets $\{A_j \mid j \in J\}$ such that $(\mathcal{S} \cap B_2) \setminus K \subset \cup_{j \in J} A_j$, and for every $j \in J$, there exists $i \in I$ such that $A_j \subset H_i^+$. So, $\cup_{j \in J} A_j \subset \cup_{i \in I} H_i^+$. As $H_i^+ \subset (\mathcal{S} \cap B_2) \setminus K \subset \cup_{j \in J} A_j$ for every $i \in I$, we have $\cup_{i \in I} H_i^+ \subset \cup_{j \in J} A_j$. Therefore, $\cup_{j \in J} A_j = \cup_{i \in I} H_i^+$. Since $\{A_j \mid j \in J\}$ is locally finite, we have $\overline{\cup_{j \in J} A_j} = \cup_{j \in J} \overline{A_j}$.[15] So, $\operatorname{Fr} K = K \cap \overline{\cup_{i \in I} H_i^+} = K \cap \overline{\cup_{j \in J} A_j} = K \cap (\cup_{j \in J} \overline{A_j})$.

5. Consider $x \in \operatorname{Fr} K$. As $x \in \operatorname{Fr} K \subset K = \cap_{i \in I} H_i^-$, we have $h_i(x) \leq c_i$ for every $i \in I$. As $x \in \operatorname{Fr} K = K \cap (\cup_{j \in J} \overline{A_j})$, we have $x \in \overline{A_j}$ for some $j \in J$. Therefore, $x \in \overline{A_j} \subset \overline{H_k^+}$ for some $k \in I$. As h_k is continuous, $h_k(x) \geq c_k$, and so, $h_k(x) = c_k \geq 0$.

6. Let $K' = \{x_n \mid n \in \mathcal{N}\}$ and $z_n = \sum_{k=1}^{n} 2^{-k} x_k$ for $n \in \mathcal{N}$. Then, $z_n \in K$ as $K' \subset K$, $0 \in K$, and K is convex. As K is a compact metric space, the sequence (z_n) has a cluster point $\bar{z} \in K$.

7. Suppose $\bar{z} \in \operatorname{Fr} K$. By step 5, $h_i(\bar{z}) = c_i \geq 0$ for some $i \in I$. As $K' \subset K$, $h_i \leq c_i$ on K'. Suppose $h_i = c_i$ on K'. As K' is dense in K and h_i is continuous, $h_i = c_i$ on K. As $0 \in K$, we have $c_i = h_i(0) = 0$. So, $h_i = 0$ on K. As h_i is linear, $h_i = 0$ on the linear span of K. As h_i is continuous, $h_i = 0$ on $\mathcal{S}$, which contradicts the assumption that h_i is non-zero on $\mathcal{S}$. So, there exists $n \in \mathcal{N}$ and $\epsilon > 0$ such that $h_i(x_n) = c_i - \epsilon$. By definition, $\bar{z} = \lim_{m \uparrow \infty} z_{n_m} = \lim_{m \uparrow \infty} \sum_{k=1}^{n_m} 2^{-k} x_k$ for some strictly increasing sequence $(n_m) \subset \mathcal{N}$. As (n_m) is strictly increasing in m, there exists $M \in \mathcal{N}$ such that $n_m > n$ for $m > M$. As h_i is linear, $m \in \mathcal{N}$ and $m > M$ implies $h_i(z_{n_m}) = h_i\left(\sum_{k=1}^{n_m} 2^{-k} x_k\right) = \sum_{k=1}^{n_m} 2^{-k} h_i(x_k) \leq \sum_{k=1}^{n_m} 2^{-k} c_i - 2^{-n}\epsilon$. As h_i is continuous, $c_i = h_i(\bar{z}) = h_i(\lim_{m \uparrow \infty} z_{n_m}) = \lim_{m \uparrow \infty} h_i(z_{n_m}) \leq c_i - 2^{-n}\epsilon < c_i$, which is a contradiction. Therefore, $\bar{z} \in \operatorname{Int} K$. ■

Proof of Lemma 5.20(A) Consider K, $\mathcal{K}$, and f *ex hypothesi*.

1. H_f is closed and convex. As the proof of Theorem 5.18 shows that K is convex, compact, and metrisable, $K \cap H_f$ also has these properties.

2. Let $D : C_i \to \Delta(C_i)$ map $c \in C_i$ to the Dirac measure $\delta_c \in \Delta(C_i)$. As D is an embedding (Parthasarathy [9], Lemma II.6.1), $D(C_i)$ is a compact metric space.

[15] For every $k \in J$, as $A_k \subset \cup_{j \in J} A_j$, we have $\overline{A_k} \subset \overline{\cup_{j \in J} A_j}$. So, $\cup_{j \in J} \overline{A_j} \subset \overline{\cup_{j \in J} A_j}$. For the converse, it is sufficient to show that $\cup_{j \in J} \overline{A_j}$ is a closed set. Consider $x \notin \cup_{j \in J} \overline{A_j}$. By local finiteness, there exists an open neighbourhood U of x and a finite set $J' \subset J$ such that $U \cap A_j = \emptyset$ for every $j \in J \setminus J'$. So, $V = U \setminus (\cup_{j \in J'} \overline{A_j})$ is an open neighbourhood of x and $V \cap A_j = \emptyset$ for every $j \in J$. Therefore, $V \cap \overline{A_j} = \emptyset$ for every $j \in J$, i.e., $V \cap (\cup_{j \in J} \overline{A_j}) = \emptyset$. So, the complement of $\cup_{j \in J} \overline{A_j}$ is open and $\cup_{j \in J} \overline{A_j}$ is closed.

Evidently, δ_c is an extreme point of $\Delta(C_i)$ for every $c \in C_i$. Conversely, suppose z is an extreme point of $\Delta(C_i)$. If $z(A) \in (0,1)$ for some $A \in \mathcal{B}(C_i)$, then define $x, y \in \Delta(C_i)$ by $x(.) = z(. \cap A)/z(A)$ and $y(.) = z(. \setminus A)/[1 - z(A)]$. As $z = z(A)x + (1 - z(A))y$, z is not an extreme point of $\Delta(C_i)$, which is a contradiction. So, $z(A) \in \{0, 1\}$ for every $A \in \mathcal{B}(C_i)$. Suppose $c, c' \in \operatorname{supp} z$ and $c \neq c'$. As C_i is Hausdorff, there exist disjoint open neighbourhoods U and V of c and c' respectively. As $c, c' \in \operatorname{supp} z$, we have $z(U) = 1 = z(V)$, which contradicts $z \in \Delta(C_i)$. So, $\operatorname{supp} z$ is a singleton. Therefore, $D(C_i)$ is the set of extreme points of $\Delta(C_i)$.

Setting $E = D(C_i) - \{\alpha\}$, it is clear that $E \subset K$ and E is the compact metric space of extreme points of K.

3. Let $E_0 = E \cap K \cap H_f$. As $E \subset K$, Theorem II.10.3 in Schaefer [12] implies $E_0 = E \cap K \cap H_f = E \cap H_f \neq \emptyset$. Clearly, E_0 is a compact metric space.

   If $x \in E_0$, then x is an extreme point of $K \cap H_f$. Conversely, suppose x is an extreme point of $K \cap H_f$ and $x \notin E_0$. So, $x \in (K \cap H_f) \setminus E$. Then, $f(x) = \max f(K)$ and $x = ty + (1-t)z$ for some $t \in (0, 1)$ and $y, z \in K$. It follows that $\max f(K) = f(x) = tf(y) + (1-t)f(z)$. If $f(y) < \max f(K)$, then $f(z) > \max f(K)$, which contradicts $z \in K$. So, $f(y) = f(z) = \max f(K)$, i.e., $y, z \in K \cap H_f$. Therefore, x is not an extreme point of $K \cap H_f$, which is a contradiction. So, an extreme point of $K \cap H_f$ must belong to E_0. Thus, E_0 is the set of extreme points of $K \cap H_f$.

4. Let F be the translation $x \mapsto x + \alpha$ on $\operatorname{rca}(C_i)$. As F is an affine homeomorphism, step 1 implies that $F(K \cap H_f)$ is a convex, compact, and metrisable subset of $\Delta(C_i)$, with $F(E_0)$ as the set of extreme points. By the Krein-Milman theorem (Schaefer [12], Theorem II.10.4), $F(K \cap H_f) = \overline{\operatorname{co}}\, F(E_0)$.

5. By Lemma V.2.4 in Dunford and Schwartz [7], $\overline{\operatorname{co}}\, F(E_0) = \overline{\operatorname{co} F(E_0)}$. As $E_0 \subset E$, we have $F(E_0) \subset F(E) = E + \{\alpha\} = D(C_i)$. Let $C_i^0 = D^{-1} \circ F(E_0)$. So, $C_i^0 \subset C_i$. Using step 3, as $D^{-1} : D(C_i) \to C_i$ and F are homeomorphisms, C_i^0 is a compact metric subset of C_i. So, $\Delta(C_i^0)$ is a compact metric space (Parthasarathy [9], Theorem II.6.4) and $\overline{\operatorname{co} F(E_0)} = \overline{\operatorname{co} D(C_i^0)} \subset \Delta(C_i^0)$. As C_i^0 is separable metric, $\overline{\operatorname{co} D(C_i^0)} \supset \Delta(C_i^0)$ (Parthasarathy [9], Theorem II.6.3). We conclude that $F(K \cap H_f) = \overline{\operatorname{co}}\, F(E_0) = \Delta(C_i^0)$, i.e., $K \cap H_f = \Delta(C_i^0) - \{\alpha\}$.

6. Let $\gamma \in \Delta(C_i^0)$. As A_f is the closed affine span of $K \cap H_f$, $A_f + \{\alpha - \gamma\}$ is the closed affine span of $(K \cap H_f) + \{\alpha - \gamma\} = \Delta(C_i^0) - \{\gamma\}$. Since $\gamma \in \Delta(C_i^0)$, $A_f + \{\alpha - \gamma\}$ is the closed linear span of $\Delta(C_i^0) - \{\gamma\}$.

By Lemma 5.17, $\text{Int}\,[\Delta(C_i^0) - \{\gamma\}] \neq \emptyset$ relative to $A_f + \{\alpha - \gamma\}$. Using step 5, $\text{Int}\,(K \cap H_f) = \text{Int}\,[\Delta(C_i^0) - \{\alpha\}] \neq \emptyset$ relative to A_f. ■

Proof of Lemma 5.20(B) Consider K, $\mathcal{K}$, and f *ex hypothesi*.

1. Suppose $x_0 \in \text{Int}\,(K \cap H_f)$ relative to A_f, $H_f = A_f$, and $x_0 \notin \text{Int}\,(K \cap H_f)$ relative to $\text{Fr}\,K$. We derive a contradiction.

2. As $x_0 \notin \text{Int}\,(K \cap H_f)$ relative to $\text{Fr}\,K$, there is a sequence $(x_n) \subset \text{Fr}\,K \setminus (K \cap H_f)$ converging to x_0. As $\text{Fr}\,K \subset K$, we have $\text{Fr}\,K \setminus (K \cap H_f) = \text{Fr}\,K \setminus H_f$. So, $(x_n) \subset \text{Fr}\,K \setminus H_f$.

3. As $x_0 \in H_f$ and $x_n \in K \setminus H_f$, we have $f(x_0) > f(x_n)$ for every $n \in \mathcal{N}$. By Lemma 5.17, there exists $c \in \text{Int}\,K$. Let $\alpha = x_0 - c$. As $c \in K \setminus H_f$, we have $f(\alpha) = f(x_0) - f(c) > 0$. Let $t_n = [f(x_0) - f(x_n)]/f(\alpha) > 0$ for $n \in \mathcal{N}$. Setting $y_n = x_n + t_n\alpha$, we have $f(y_n) = f(x_n) + t_n f(\alpha) = f(x_0)$. So, $(y_n) \subset H_f$. Set $z_n = y_n - \alpha$ for $n \in \mathcal{N}$. As $\lim_n x_n = x_0$, we have $\lim_n t_n = 0$. Therefore, $\lim_n y_n = \lim_n x_n = x_0$ and $\lim_n z_n = c$.

4. Since $c \in \text{Int}\,K$, there exists an open neighbourhood U of c such that $U \subset K$. Then, $U + \{\alpha\}$ is an open neighbourhood of x_0. As $\lim_n x_n = x_0$ and $\lim_n f(x_n) = f(x_0) > f(c)$, we may assume that $(x_n) \subset U + \{\alpha\}$ and $f(x_n) > f(c)$ for every $n \in \mathcal{N}$. Then, $t_n \in (0, 1)$ for every $n \in \mathcal{N}$. As $\lim_n y_n = x_0$, there exists $n_0 \in \mathcal{N}$ such that $n > n_0$ implies $y_n \in U + \{\alpha\}$ and $z_n \in U \subset \text{Int}\,K$. So, $n > n_0$ implies $z_n \in \text{Int}\,K$, $x_n \in \text{Fr}\,K$, and $x_n = (1 - t_n)y_n + t_n z_n$. If $y_n \in K$, then $x_n \in \text{Int}\,K$ (Schaefer [12], Proposition II.1.1), which is a contradiction. Therefore, $y_n \in H_f \setminus K \subset H_f \setminus (K \cap H_f)$ for $n > n_0$.

5. Since $x_0 \in \text{Int}\,(K \cap H_f)$ relative to A_f and $A_f = H_f$, it follows that $x_0 \in \text{Int}\,(K \cap H_f)$ relative to H_f. Consequently, there exists V open in $\mathcal{K}$ such that $x_0 \in V \cap H_f \subset K \cap H_f$. Since $\lim_n y_n = x_0$, there exists $n_1 \in \mathcal{N}$ such that $n > n_1$ implies $y_n \in V \cap H_f \subset K \cap H_f$.

6. So, $n > \max\{n_0, n_1\}$ implies $y_n \in H_f \setminus (K \cap H_f)$ and $y_n \in K \cap H_f$, which is a contradiction. ■

References

[1] C. Aliprantis, K. Border: *Infinite Dimensional Analysis: A Hitchhiker's Guide, 3rd ed.*, Springer-Verlag (2006)

[2] P. Billingsley: *Convergence of Probability Measures, 2nd ed.*, John Wiley & Sons (1999)

[3] H. Brezis: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer (2011)

[4] F. Browder: The fixed point theory of multi-valued mappings in topological vector spaces, *Mathematische Annalen* 177, 283-301 (1968)

[5] E. van Damme: *Refinements of the Nash Equilibrium Concept*, Springer-Verlag, Berlin (1983)

[6] J. Dugundji: *Topology*, Wm. C. Brown (1989)

[7] N. Dunford, J. Schwartz: *Linear Operators, Part I: General Theory*, Wiley Interscience, New York (1958)

[8] D. Gale, S. Sherman: Solutions of finite two-person games, in *Contributions to the Theory of Games, Vol. 1*, (eds.) H. Kuhn, A. Tucker, Annals of Mathematical Studies 24, Princeton University Press (1950)

[9] K. Parthasarathy: *Probability Measures on Metric Spaces,* Academic Press (1967)

[10] D. Pearce: Rationalizable strategic behavior and the problem of perfection, *Econometrica* 52, 1029-1050 (1984)

[11] M. Rao: *Measure Theory and Integration*, Wiley Interscience (1987)

[12] H. Schaefer: *Topological Vector Spaces*, Springer-Verlag (1986)

[13] A. Zimper: Equivalence between best responses and undominated strategies: a generalization from finite to compact strategy sets, *Economics Bulletin* 3, 1-6 (2005)