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Abstract

We propose an asset's money-metric value as the appropriate representation of its subjective value to an investor. This value is expressed in monetary terms and is invariant across equivalent utility representations of the investor's preference. The ordering of money-metric values across assets matches the investor's preference ordering over the assets. The money-metric value of a risky asset is inversely related to the investor's risk aversion, while the money-metric value of a risk-free asset is uniform across preferences with comparable risk-aversion. Finally, an asset's arbitrage-free market price is the sum of its money-metric value and the investor's willingness-to-pay for fully de-risking the asset.

JEL classification: G11, G12

Key words: money-metric asset valuation, arbitrage-free prices, risk aversion

1 Introduction

Let an asset be represented by a probability measure over a set of outcomes. While our narrative will interpret outcomes as cash-flows over time, our model is not tied to this interpretation. An investor's (von Neumann-Morgenstern) utility defined over the outcomes generates her expected utility from an asset, which is a subjective hedonistic valuation, while the asset's market price is an objective measure of its exchange value to a price-taking investor. We propose a hybrid of these valuation methods. Given an asset and asset prices, there may be a cheaper asset that yields the investor an expected utility that meets or exceeds the expected utility from the given asset. Accordingly, the least cost of replicating or exceeding the expected utility from a given asset will be called its money-metric value.

Consider an investor whose utility u is defined over the admissible outcomes for that investor and assets whose supports are confined to these admissible outcomes. Asset μ's money-metric value to the investor is $M(\mu, u)$,

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which depends on the investor's expected utility function $U(., u)$ defined over the assets and the market prices of assets generated by a function P. Money-metric valuations have the following desirable properties:

(a) $M(\mu, u)$ assigns a monetary value to the asset, which depends on the investor's preference but is invariant across its equivalent utility representations. This monetary denomination makes $M(\mu, u)$ and $P(\mu)$ comparable.

(b) $M(\mu, u)$ relates to $P(\mu)$ in a simple, precise, and economically meaningful and revealing way. Specifically, Theorem 3.3 shows that $P(\mu)$ is the sum of $M(\mu, u)$ and the investor's willingness-to-pay for fully de-risking asset μ. An asset's riskiness refers to the random deviations of its cash-flows from its 'mean' risk-free cash-flow. Accordingly, de-risking the asset refers to the elimination of these deviations, thereby leaving the investor with the mean cash-flow.

(c) For assets μ and λ, Theorem 3.4 shows that the ordering of $M(\mu, u)$ and $M(\lambda, u)$ matches the ordering of $U(\mu, u)$ and $U(\lambda, u)$.[1]

(d) Theorem 3.5 shows that, if asset μ is risky, then $M(\mu, .)$ varies inversely with the risk aversion of the given preferences, and if μ is risk-free, then $M(\mu, .)$ is uniform for preferences with comparable risk aversion.

In contrast to properties (a) and (b) of money-metric valuations, hedonic valuations are not denominated in monetary units and therefore they cannot be compared sensibly with market prices. Unlike property (d), hedonic valuations are not meaningfully comparable across preferences since they are determinate only up to increasing affine transformations. Contrasted with properties (c) and (d), market prices are not systematically related to hedonic valuations and individual preferences respectively.

A key assumption for our results is that the given price function P is continuous and generates arbitrage-free asset prices.[2] This yields the useful facts recorded in Lemma 3.1, namely, the prices of risk-free assets generate an increasing and continuous linear functional p on the space of all potential outcomes and the price of an admissible asset is the expectation of functional p with respect to the asset's distribution.

The rest of this paper is organised as follows. Section 2 prepares the technical ground, Section 3 derives the substantive results, and Section 4 sums-up the results. The lemmas are proved in the Appendix.

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[1] For a suitable u, an appropriate stochastic dominance ordering of μ and λ implies the ordering of $U(\mu, u)$ and $U(\lambda, u)$, which in turn implies the ordering of $M(\mu, u)$ and $M(\lambda, u)$; see Strassen [14], Shaked and Shanthikumar [13], and Shah [11] for characterisations of various notions of stochastic dominance that are relevant in Section 2's setting.

[2] Arbitrage-free asset pricing theory starts with Black and Scholes [1]. Early extensions include Cox and Ross [2], Ross [9], Merton [6], and Harrison and Kreps [4]. Kreps [5] initiates the study of the relationship between equilibrium and arbitrage-free prices.

2 Technical preparation

The following specifications hold throughout this paper.

1. X is a metrisable locally convex topological vector space and X^* is the set of continuous real-valued linear functionals on X. X^* is a total family of linear functionals on X, i.e., if $x \in X$ and $h(x) = 0$ for every $h \in X^*$, then $x = 0$.[3] $\mathcal{B}(X)$ is the Borel σ-algebra on X. $\geq$ is a preordering on X. For $x, y \in X$, we say that $x > y$ if $x \geq y$ and $x \neq y$.

The leading interpretation X will be as path-spaces of random processes.

2. O is a compact subset of X with $|O| > 1$. O is given the subspace topology derived from X. The Borel σ-algebra on O, namely $\mathcal{B}(O)$, coincides with the trace σ-algebra generated on O by $\mathcal{B}(X)$, i.e., $E \in \mathcal{B}(O)$ if and only if $E = O \cap F$ for some $F \in \mathcal{B}(X)$. Since $O \in \mathcal{B}(X)$, it follows that $\mathcal{B}(O) \subset \mathcal{B}(X)$.

We interpret O as the **set of admissible outcomes** for the investor that meet some exogenous desiderata, say, bounds on cash-flows; elements in $X \setminus O$ may be admissible outcomes for other investors.

3. $\mathcal{C}(X)$ (resp., $\mathcal{C}(O)$) is the set of continuous and bounded real-valued functions defined on X (resp., O).

4. $\Delta(X)$ (resp., $\Delta(O)$) is the set of probability measures on $(X, \mathcal{B}(X))$ (resp., $(O, \mathcal{B}(O))$), which is given its weak* topology.[4] We extend $\mu \in \Delta(O)$ to $T(\mu) \in \Delta(X)$ by setting $T(\mu)(B) = \mu(B \cap O)$ for $B \in \mathcal{B}(X)$. Evidently, $T(\Delta(O)) = \{\mu \in \Delta(X) \mid \mu(O) = 1\}$, which will be represented by $\Delta(X)_O$.

5. For $\mu \in \Delta(X)$ (resp., $\mu \in \Delta(O)$), $m_\mu \in X$ is called a (Pettis, or weak) mean of μ if $x^*(m_\mu) = \int_X \mu(dz)\, x^*(z)$ (resp., $x^*(m_\mu) = \int_O \mu(dz)\, x^*(z)$) for every $x^* \in X^*$.[5] Since X^* is a total family of linear functionals on X, if $m_\mu \in X$ exists, then it is unique. Let $\Delta(X)_0$ (resp., $\Delta(O)_0$) denote the set of measures $\mu \in \Delta(X)$ (resp., $\mu \in \Delta(O)$) for which a mean $m_\mu \in X$ exists.

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[3] Consider $x \in X \setminus \{0\}$. Given X *ex hypothesi*, there exist disjoint convex open neighbourhoods A and B of 0 and x respectively. Then, there exists a closed real hyperplane H that strictly separates A and B (Schaefer [10], Chapter II, Theorem 9.1). Hence, there exists $h \in X^*$ and $\alpha \in \Re$ such that $H = h^{-1}(\{\alpha\})$ (Schaefer [10]), Chapter I, Proposition 4.2). So, either $h(x) > \alpha > h(0) = 0$ or $h(x) < \alpha < h(0) = 0$.

[4] This is the weakest topology on $\Delta(X)$ that renders the functionals $\Delta(X) \ni \mu \mapsto \int_X \mu(dx)\, f(x) \in \Re$ continuous for every $f \in \mathcal{C}(X)$. The weak* topology on $\Delta(O)$ is defined analogously.

[5] Vector m_μ is the weak integral of the identity function *à la* Pettis [8]. If $X = \Re^n$, then $m_\mu \in \Re^n$ is characterised by the equations $\langle e_i, m_\mu \rangle = \int_X \mu(dz)\, \langle e_i, z \rangle$ for the canonical basis $\{e_1, \ldots, e_n\} \subset \Re^n$, which amounts to computing m_μ component-by-component.

An **asset** refers to $\mu \in \Delta(X)_0$.[6] Asset μ is admissible for the investor if $\mu \in \Delta(X)_O$. If X is a path-space – say, a set of possible cash-flows over a time domain – then it can be shown very generally that every asset $\mu \in \Delta(X)_0$ specifies a random process whose possible outcomes are the cash-flows in X. Using this fact, it can be shown that distributions on the above-specified X generate a large family of random processes that are commonly used in financial modelling.[7] We now collect some consequences of the above specifications.

Lemma 2.1 *Consider X and O as specified above.*

(A) $\Delta(O)$ *is compact metric.*

(B) $T : \Delta(O) \to \Delta(X)_O$ *is a homeomorphism, with* $T^{-1}(\mu)$ *being the restriction of* $\mu \in \Delta(X)_O$ *to* $\mathcal{B}(O)$.

(C) $\Delta(X)_O$ *is compact metric.*

(D) If $\mu \in \Delta(O)_0$*, then* $T(\mu) \in \Delta(X)_0 \cap \Delta(X)_O$ *and* $m_{T(\mu)} = m_\mu$.

(E) If $\mu \in \Delta(X)_0 \cap \Delta(X)_O$*, then* $T^{-1}(\mu) \in \Delta(O)_0$ *and* $m_{T^{-1}(\mu)} = m_\mu$.

(F) If O *is convex and* $\mu \in \Delta(X)_O$*, then* $\mu \in \Delta(X)_0$ *and* $m_\mu \in O$.

3 Results

Asset $\mu \in \Delta(X)_0$ is said to be **risk-free** if $\mu = \delta_x$ for some $x \in X$, where δ_x is the Dirac measure at x. Given the interpretation of outcomes as cash-flows, the risk-free asset δ_x delivers the cash-flow x with probability one. While the cash delivered by x can vary over time, the variations are completely predictable; hence, δ_x is risk-free.

An **asset portfolio** is a function $\theta : \Delta(X)_0 \to \Re$ such that $\operatorname{supp}\theta = \theta^{-1}(\Re\setminus\{0\})$ is finite. $\theta(\mu) > 0$ (resp., $\theta(\mu) < 0$) refers to the purchase (resp., sale) of $|\theta(\mu)|$ units of asset μ.

Asset prices are exogenously given by $P : \Delta(X)_0 \to \Re$. P is said to be **expected-arbitrage-free** if there is no portfolio θ such that

$$\sum_{\mu\in\operatorname{supp}\theta} \theta(\mu)P(\mu) \leq 0 \qquad \text{and} \qquad \sum_{\mu\in\operatorname{supp}\theta} \theta(\mu)m_\mu \geq 0 \tag{1}$$

[6]An asset μ is equivalent to all portfolios of assets that generate the distribution μ.

[7]While these facts illustrate the model's wide scope, they are not required for the results in this paper. Therefore, we refer the interested reader to Appendices A and B in Shah [11] for demonstrations of the facts that the accommodated processes include Wiener process, Brownian motion, absorbed Brownian motion, geometric Brownian motion, Ornstein-Uhlenbeck process, and a large class of second-order processes.

with either inequality strict. Define the **outcome price functional** $p : X \to \Re$ by $p(x) = P(\delta_x)$. The following result establishes the key properties of p and P.

Lemma 3.1 *Consider X and O as specified in Section 2. If $P : \Delta(X)_0 \to \Re$ is continuous and expected-arbitrage-free, then*

(A) $p \in X^*$, p is increasing, and

(B) if O is convex, then $\Delta(X)_O \subset \Delta(X)_0$ and $P(\mu) = \int_O \mu(dx)\, p(x) = p(m_\mu)$ for every $\mu \in \Delta(X)_O$.

Suppose O is convex. By Lemma 2.1, $m_\mu \in O$ for every admissible asset $\mu \in \Delta(X)_O$. We say that $u : O \to \Re$ is **risk averse** if $u(m_\mu) \geq \int_O \mu(dy)\, u(y)$ for every $\mu \in \Delta(X)_O$, i.e., the risk-free cash-flow m_μ is preferred to the risky cash-flow generated by μ. The following assumptions hold henceforth.

Assumption 3.2 *In addition to the specifications of Section 2, suppose*

(a) O is convex and there exists $l \in O$ such that $x \geq l$ for every $x \in O$,

(b) $u : O \to \Re$ and $v : O \to \Re$ are continuous, risk averse, non-constant, increasing, and

(c) $P : \Delta(X)_0 \to \Re$ is continuous and expected-arbitrage-free.

As O is compact, $u \in \mathcal{C}(O)$. Hence, the **expected utility** from $\mu \in \Delta(O)$ is $U(\mu, u) = \int_O \mu(dy)\, u(y) \in \Re$. $U(., u) : \Delta(O) \to \Re$ is continuous as $u \in \mathcal{C}(O)$ and $\Delta(O)$ has the weak* topology. Given an admissible asset $\mu \in \Delta(X)_O$ and u, let $V(\mu, u) = U(T^{-1}(\mu), u)$. As T^{-1} is continuous by Lemma 2.1, $V(., u) : \Delta(X)_O \to \Re$ is continuous.

The **money-metric value** of an admissible asset $\mu \in \Delta(X)_O$ to the investor with utility u is

$$M(\mu, u) = \inf\{P(\lambda) \mid \lambda \in \Delta(X)_O,\ V(\lambda, u) \geq V(\mu, u)\} \tag{2}$$

i.e., it is the price of the cheapest asset that is admissible and yields at least as much expected utility as μ. It is immediate that, if $v : O \to \Re$ is an increasing affine transformation of u, then $M(\mu, v) = M(\mu, u)$, i.e., the money-metric value of an asset depends on the preference, but not on a particular von Neumann-Morgenstern representation of the preference. Also note that, as $V(., u)$ is continuous, Lemma 2.1 implies $\{\lambda \in \Delta(X)_O \mid V(\lambda, u) \geq V(\mu, u)\}$ is compact. As P is continuous, there exists $\lambda \in \Delta(X)_O$ such that $U(\lambda, u) \geq U(\mu, u)$ and $M(\mu, u) = P(\lambda)$.

Given an admissible asset $\mu \in \Delta(X)_O$ and u, the set of admissible **certainty equivalent outcomes** is

$$E(\mu, u) = \{x \in O \mid u(x) = V(\mu, u)\} \tag{3}$$

and the set of **risk premia** is

$$\Pi(\mu, u) = \{\pi \in X \mid m_\mu - \pi \in O,\ u(m_\mu - \pi) = V(\mu, u)\} \tag{4}$$

Clearly, $x \in E(\mu, u)$ if and only if $m_\mu - x \in \Pi(\mu, u)$. Given μ, a risk premium $\pi \in \Pi(\mu, u)$ is a maximal cash-flow that the investor is willing to sacrifice in order to get the risk-free net cash-flow $m_\mu - \pi$ instead of the hedonically equivalent risky cash-flow generated by μ. As cash-flows are vectors, there are multiple certainty equivalent outcomes and multiple risk premia associated with an asset. While X's zero-vector is necessarily a risk premium for a risk-free asset, there can be other risk premia.

Since $\pi \in \Pi(\mu, u)$ is a cash-flow 'payment' for swapping the risky asset μ with its risk-free mean cash-flow m_μ, the investor's willingness-to-pay for this exchange is measured by the maximal market value over the set of risk premia, i.e., $\max p \circ \Pi(\mu, u)$.

Part (A) of the following result characterises an asset's money-metric value in terms of the prices of its certainty equivalent assets. Part (B) is the promised decomposition of an asset's market price.

Theorem 3.3 *Given Assumption 3.2, if $\mu \in \Delta(X)_O$, then*

(A) $M(\mu, u) = \min p \circ E(\mu, u)$, and

(B) $P(\mu) - M(\mu, u) = \max p \circ \Pi(\mu, u) \geq 0$.

Proof. As O is a compact subset of the metric space X, it is closed in X. As O is also convex, $u(O)$ is a closed interval. Hence, $V(\mu, u) \in u(O)$. So, $E(\mu, u) \neq \emptyset$. As u is continuous, it easily follows that $E(\mu, u)$ is closed in X. As O is compact and $E(\mu, u) \subset O$, $E(\mu, u)$ is compact.

(A) Consider $x \in E(\mu, u)$. Then, $\delta_x \in \Delta(X)_O$ and $V(\delta_x, u) = u(x) = V(\mu, u)$. Hence, $M(\mu, u) \leq P(\delta_x) = p(x)$. As this holds for every $x \in E(\mu, u)$, we have $M(\mu, u) \leq \inf p \circ E(\mu, u)$.

We have $M(\mu, u) = P(\lambda)$ for some $\lambda \in \Delta(X)_O$ such that $V(\lambda, u) \geq V(\mu, u)$. Lemma 2.1 implies $m_\lambda \in O$. As u is risk averse, $u(m_\lambda) = u(m_{T^{-1}(\lambda)}) \geq U(T^{-1}(\lambda), u) = V(\lambda, u) \geq V(\mu, u) \geq u(l)$. As u is continuous and O is convex, there exists $t \in [0, 1]$ such that $tm_\lambda + (1-t)l \in E(\mu, u)$. Since $m_\lambda \geq l$, Lemma 3.1 implies $p(m_\lambda) \geq p(l)$ and $\inf p \circ E(\mu, u) \leq p(tm_\lambda + (1-t)l) = tp(m_\lambda) + (1-t)p(l) \leq p(m_\lambda) = P(\lambda) = M(\mu, u)$.

Hence, $M(\mu, u) = \inf p \circ E(\mu, u)$. Since $E(\mu, u)$ is compact and p is continuous, $\inf p \circ E(\mu, u) = \min p \circ E(\mu, u)$.

(B) By (A), there exists $x \in E(\mu, u)$ such that $p(x) = M(\mu, u)$. Setting $g_\mu = m_\mu - x \in \Pi(\mu, u)$, we have $p(m_\mu - g_\mu) = p(x) = M(\mu, u)$.

Suppose $h_\mu \in \Pi(\mu, u)$ and $p(h_\mu) > p(g_\mu)$. Then, $m_\mu - h_\mu \in E(\mu, u)$ and $p(m_\mu - h_\mu) = p(m_\mu) - p(h_\mu) < p(m_\mu) - p(g_\mu) = p(m_\mu - g_\mu) = M(\mu, u) = \min p \circ E(\mu, u)$, a contradiction. So, $p(g_\mu) = \max p \circ \Pi(\mu, u)$.

So, Lemma 3.1 implies $P(\mu) = p(m_\mu) = p(g_\mu + x) = p(g_\mu) + p(x) = \max p \circ \Pi(\mu, u) + M(\mu, u)$. As $P(\mu) \geq M(\mu, u)$, we have $\max p \circ \Pi(\mu, u) \geq 0$. ■

The next result shows that the ordering of an investor's money-metric valuations of admissible assets mimics her hedonic preference over the assets.

Theorem 3.4 *Given Assumption 3.2, if* $\mu, \lambda \in \Delta(X)_O$*, then* $V(\mu, u) \leq V(\lambda, u)$ *if and only if* $M(\mu, u) \leq M(\lambda, u)$.

Proof. It follows from (2) that $V(\mu, u) \leq V(\lambda, u)$ implies $M(\mu, u) \leq M(\lambda, u)$.

Conversely, suppose $M(\mu, u) \leq M(\lambda, u)$ and $V(\mu, u) > V(\lambda, u)$. It follows from (2) that $M(\mu, u) \geq M(\lambda, u)$, and therefore, $M(\mu, u) = M(\lambda, u)$.

Theorem 3.3 implies the existence of $x \in E(\mu, u)$ such that $p(x) = M(\mu, u) = M(\lambda, u)$. Since O is convex, $u(x) = V(\mu, u) > V(\lambda, u) \geq u(l)$, and u is continuous, there exists $t \in [0, 1)$ such that $tx + (1-t)l \in E(\lambda, u)$.

As $u(x) > u(l)$, we have $x \neq l$. Therefore, $x > l$. Applying Lemma 3.1 and Theorem 3.3, $p(x) > p(l)$ and $p(tx + (1-t)l) = tp(x) + (1-t)p(l) < p(x) = M(\lambda, u) = \min p \circ E(\lambda, u) \leq p(tx + (1-t)l)$, a contradiction. ■

Since $V(., u)$ is continuous on the compact set $\Delta(X)_O$, there exists an optimal asset, i.e., $\mu \in \Delta(X)_O$ such that $V(\mu, u) \geq V(\lambda, u)$ for every $\lambda \in \Delta(X)_O$. Applying Theorem 3.4, the optimal asset maximises the money-metric value among the assets in $\Delta(X)_O$. Applying Theorem 3.3, the optimal asset also maximises the difference between its market price and the investor's willingness-to-pay to de-risk the asset's cash-flow.

Finally, we show that money-metric valuations and risk aversion are inversely related. Given $x \in O$ and utility u satisfying Assumption 3.2, let

$$A(x, u) = \{\mu \in \Delta(X)_O \mid u(x) \leq V(\mu, u)\}$$

be the **acceptance set** (Yaari [15]) of u at x. Given utilities u and v that satisfy Assumption 3.2, we say that u is **more risk averse** than v if $A(x, u) \subset A(x, v)$ for every $x \in O$. The notion motivating this definition is that a 'less risk averse' utility will accept every risk that is accepted by the 'more risk averse' utility.[8]

[8] In our vector outcomes setting, this criterion is equivalent to the natural extensions of several other criteria for comparing risk aversion in the setting with real outcomes (Shah [11], Theorems 4.5 and 5.3).

Theorem 3.5 *In addition to Assumption 3.2, suppose for all $x, y \in O$, there exists $z \in O$ such that $z \geq x$ and $z \geq y$. If u is more risk averse than v, then*

(A) $M(\mu, u) \leq M(\mu, v)$ for every $\mu \in \Delta(X)_O$, and

(B) $M(\delta_x, u) = M(\delta_x, v)$ for every $x \in O$.

Proof. Consider X, O, u, and v *ex hypothesi*. We start with an observation.

- $\neg \pi_v > \pi_u$ *for all* $\pi_u \in \Pi(\mu, u)$ *and* $\pi_v \in \Pi(\mu, v)$.

  Suppose $\pi_v > \pi_u$ for some $\pi_u \in \Pi(\mu, u)$ and $\pi_v \in \Pi(\mu, v)$. Let $v^*(.) = v(.) + u(m_\mu - \pi_u) - v(m_\mu - \pi_u)$. Then, $A(., v^*) = A(., v)$, $\Pi(., v^*) = \Pi(., v)$, $E(., v^*) = E(., v)$, $u(m_\mu - \pi_u) = v^*(m_\mu - \pi_u)$, and $m_\mu - \pi_v \in E(\mu, v) = E(\mu, v^*)$. So, $V(\mu, v^*) = v^*(m_\mu - \pi_v) < v^*(m_\mu - \pi_u) = u(m_\mu - \pi_u) = V(\mu, u)$. Thus, $\mu \in A(m_\mu - \pi_u, u) \setminus A(m_\mu - \pi_u, v^*) = A(m_\mu - \pi_u, u) \setminus A(m_\mu - \pi_u, v)$, which contradicts the hypothesis that u is more risk averse than v, i.e., $A(m_\mu - \pi_u, u) \subset A(m_\mu - \pi_u, v)$.

Now we prove the claims.

(A) Consider $\mu \in \Delta(X)_O$, $x \in E(\mu, v)$, and $y \in E(\mu, u)$.

Suppose $u(x) < u(y)$. As O has the upper bound property, there exists $z \in O$ such that $z \geq x$ and $z \geq y$. As u is increasing, $u(z) \geq u(y) > u(x)$. So, $z > x$. As $\{x + t(z - x) \mid t \in [0, 1]\}$ is convex and u is continuous, there exists $t \in (0, 1]$ such that $r = t(z - x) > 0$ and $u(x+r) = u(y)$. It follows that $m_\mu - x \in \Pi(\mu, v)$, $m_\mu - x - r \in \Pi(\mu, u)$, and $m_\mu - x > m_\mu - x - r$, which contradicts the above observation.

So, $u(x) \geq u(y) \geq u(l)$. As u is continuous, there exists $t \in [0, 1]$ such that $u(tx + (1 - t)l) = u(y)$, i.e., $tx + (1 - t)l \in E(\mu, u)$. As p is linear and increasing, $p(x) \geq p(tx + (1 - t)l) \geq \min p \circ E(\mu, u)$. Since this holds for every $x \in E(\mu, v)$, we have $M(\mu, v) = \min p \circ E(\mu, v) \geq \min p \circ E(\mu, u) = M(\mu, u)$.

(B) Fix $x \in O$. By (A), $M(\delta_x, u) \leq M(\delta_x, v)$. By definition, $M(\delta_x, u) = \inf\{P(\lambda) \mid \lambda \in \Delta(X)_O \text{ and } V(\lambda, u) \geq u(x)\} = \inf\{P(\lambda) \mid \lambda \in A(x, u)\} = \inf P \circ A(x, u)$. Similarly, $M(\delta_x, v) = \inf P \circ A(x, v)$. As $A(x, u) \subset A(x, v)$, we have $M(\delta_x, u) = \inf P \circ A(x, u) \geq \inf P \circ A(x, v) = M(\delta_x, v)$. Thus, $M(\delta_x, u) = M(\delta_x, v)$. ■

The following lemma provides an alternative route to part (B), which clarifies how the comparability of risk aversion is important for this result.

Lemma 3.6 *Given Assumption 3.2, if u is more risk averse than v, then $u(z) \geq u(y)$ if and only if $v(z) \geq v(y)$ for all $y, z \in O$.*

Thus, von Neumann-Morgenstern utilities with comparable risk aversion must be ordinally congruent.[9] If u is more risk averse than v and $\delta_x \in \Delta(X)_O$ is a risk-free asset, then $E(\delta_x, u) = E(\delta_x, v)$. Hence, Theorem 3.3 implies $M(\delta_x, u) = M(\delta_x, v)$.

Combining Theorems 3.3 and 3.5, an investor's subjective decomposition of an asset's market price varies systematically with risk aversion: as risk aversion increases, the money-metric value component of the price decreases and it is offset exactly by a higher willingness-to-pay for de-risking the asset. In the case of a risk-free asset, the decomposition in Theorem 3.3 is invariant with respect to preferences with *comparable* risk aversion.

4 Concluding remarks

We have shown that an asset's money-metric value is the appropriate way of representing its subjective value to an investor. The grounds for this claim are: (1) Equation (2) implies that it is a monetary value that is invariant across equivalent utility representations of the investor's preference, (2) an asset's market price is shown to be the sum of its money-metric value and the investor's willingness-to-pay for fully de-risking the asset (Theorem 3.3), (3) the ordering of money-metric values across assets matches the investor's preference ordering over the assets (Theorem 3.4), and (4) money-metric values are meaningfully related to an investor's attitude to risk insofar as the money-metric value of a risky asset is inversely related to the investor's risk aversion, while the money-metric value of a risk-free asset is uniform across preferences with comparable risk-averseness (Theorem 3.5).

Appendix

Proof of Lemma 2.1 Consider X and O *ex hypothesi.*

(A) As O is compact metric, $\Delta(O)$ is compact metric (Parthasarathy [7], Chapter II, Theorem 6.4).

(B) Since $\mathcal{B}(O) \subset \mathcal{B}(X)$, it is easily checked that T is bijective.

Consider a sequence $(\mu_n) \subset \Delta(O)$ converging to $\mu \in \Delta(O)$ and $f \in \mathcal{C}(X)$. Since $\int_X T(\mu_n)(dz)\, f(z) = \int_O \mu_n(dz)\, f(z) \to \int_O \mu(dz)\, f(z) = \int_X T(\mu)(dz)\, f(z)$, the sequence $(T(\mu_n)) \subset \Delta(X)$ converges to $T(\mu)$. So, T is continuous.

Consider a net $(\mu_i) \subset \Delta(X)_O$ converging to $\mu \in \Delta(X)_O$ and $f \in \mathcal{C}(O)$. As X is a metric space, it is a normal space (Dugundji [3], Chapter IX, Theorem 5.2). As O is a compact subset of a metric space, it is closed in X. By the Tietze extension theorem (Dugundji [3], Chapter VII,

[9] Ordinal congruence holds trivially for *all* increasing utilities defined over real outcomes.

Theorem 5.1), f has an extension $g \in \mathcal{C}(X)$. As $\int_O T^{-1}(\mu_i)(dz)\, f(z) = \int_X \mu_i(dz)\, g(z) \to \int_X \mu(dz)\, g(z) = \int_O T^{-1}(\mu)(dz)\, f(z)$, it follows that the net $(T^{-1}(\mu_i)) \subset \Delta(O)$ converges to $T^{-1}(\mu) \in \Delta(O)$. So, T^{-1} is continuous.

(C) follows from (A) and (B).

(D) If $\mu \in \Delta(O)_0$, then $T(\mu) \in \Delta(X)_O$ and there exists $m_\mu \in X$ such that $\int_X T(\mu)(dz)\, x^*(z) = \int_O \mu(dz)\, x^*(z) = x^*(m_\mu)$ for every $x^* \in X^*$. So, $T(\mu) \in \Delta(X)_0$ and $m_{T(\mu)} = m_\mu$.

(E) If $\mu \in \Delta(X)_0 \cap \Delta(X)_O$, then $T^{-1}(\mu) \in \Delta(O)$ and there is $m_\mu \in X$ such that $x^*(m_\mu) = \int_X \mu(dz)\, x^*(z) = \int_O \mu(dz)\, x^*(z) = \int_O T^{-1}(\mu)(dz)\, x^*(z)$ for every $x^* \in X^*$. So, $T^{-1}(\mu) \in \Delta(O)_0$ and $m_{T^{-1}(\mu)} = m_\mu$.

(F) Give $\Re^{X^*}$ the product topology and define $H : X \to \Re^{X^*}$ by $H(x) = (h(x))_{h \in X^*}$. H is continuous as every $h \in X^*$ is continuous. If $H(x) = 0$ for some $x \in X$, then $h(x) = 0$ for every $h \in X^*$, which implies $x = 0$. So, H is injective. As O is compact and $\Re^{X^*}$ is Hausdorff, H embeds O in $\Re^{X^*}$.

As O is compact and H is continuous, $H(O)$ is compact. Since $\Re^{X^*}$ is Hausdorff, $H(O)$ is closed in $\Re^{X^*}$. Moreover, $H(O)$ is metrisable.

Consider $\lambda \in \Delta(O)$ with $|\operatorname{supp} \lambda| < \infty$. For every $h \in X^*$, we have $\int_O \lambda(dz)\, h(z) = \sum_{z \in \operatorname{supp} \lambda} \lambda(\{z\}) h(z) = h(\sum_{z \in \operatorname{supp} \lambda} \lambda(\{z\}) z)$. Therefore, $m_\lambda = \sum_{z \in \operatorname{supp} \lambda} \lambda(\{z\}) z$. As O is convex and $\operatorname{supp} \lambda \subset O$, we have $m_\lambda \in O$.

Consider $\mu \in \Delta(X)_O$. Then, $T^{-1}(\mu) \in \Delta(O)$. As O is compact metric, it is separable. Consequently, there is a sequence $(\mu_n) \subset \Delta(O)$ converging to $T^{-1}(\mu)$ such that $|\operatorname{supp} \mu_n| < \infty$ for every $n \in \mathcal{N}$ (Parthasarathy [7], Chapter II, Theorem 6.3). As shown above, m_{μ_n} exists, $m_{\mu_n} \in O$ and $H(m_{\mu_n}) \in H(O)$ for every $n \in \mathcal{N}$. As the restriction of $h \in X^*$ to O is continuous and bounded, $\lim_{n \uparrow \infty} h(m_{\mu_n}) = \lim_{n \uparrow \infty} \int_O \mu_n(dz)\, h(z) = \int_O T^{-1}(\mu)(dz)\, h(z) = \int_X \mu(dz)\, h(z)$ for every $h \in X^*$. So, $\lim_{n \uparrow \infty} H(m_{\mu_n}) = (\int_X \mu(dz)\, h(z))_{h \in X^*}$.

As $(H(m_{\mu_n})) \subset H(O)$ and $H(O)$ is closed in $\Re^{X^*}$, it follows that $(\int_X \mu(dz)\, h(z))_{h \in X^*} \in H(O)$. As H embeds O in $\Re^{X^*}$, there is a unique $m_\mu \in O$ such that $H(m_\mu) = (\int_X \mu(dz)\, h(z))_{h \in X^*}$, i.e., $h(m_\mu) = \int_X \mu(dz)\, h(z)$ for every $h \in X^*$. ■

Proof of Lemma 3.1 Consider X, O, and P *ex hypothesi.*

(A) Suppose p is not linear. Then there exist $x, y \in X$ and $\alpha, \beta \in \Re$ such that $p(\alpha x + \beta y) < \alpha p(x) + \beta p(y)$. Consider the portfolio $\theta = 1_{\delta_{\alpha x + \beta y}} - \alpha 1_{\delta_x} - \beta 1_{\delta_y}$. Then, $\sum_{\mu \in \operatorname{supp} \theta} \theta(\mu) P(\mu) = P(\delta_{\alpha x + \beta y}) - \alpha P(\delta_x) -$

$\beta P(\delta_y) = p(\alpha x + \beta y) - \alpha p(x) - \beta p(y) < 0$ and $\sum_{\mu \in \operatorname{supp} \theta} \theta(\mu) m_\mu = \alpha x + \beta y - \alpha x - \beta y = 0$, which contradicts condition (1).

Suppose p is not increasing. Then there exists $x \in X$ such that $x > 0$ and $p(x) \leq 0$. If $\theta = 1_{\delta_x}$, then $\sum_{\mu \in \operatorname{supp} \theta} \theta(\mu) P(\mu) = P(\delta_x) = p(x) \leq 0$ and $\sum_{\mu \in \operatorname{supp} \theta} \theta(\mu) m_\mu = x > 0$, which contradicts condition (1).

Consider a sequence $(x_n) \subset X$ converging to $x \in X$. Then, $(\delta_{x_n}) \subset \Delta(X)_0$, $\delta_x \in \Delta(X)_0$, and $\lim_n \int_X \delta_{x_n}(dy)\, f(y) = \lim_n f(x_n) = f(x) = \int_X \delta_x(dy)\, f(y)$ for $f \in \mathcal{C}(X)$. So, (δ_{x_n}) converges to δ_x. As P is continuous, $\lim_n p(x_n) = \lim_n P(\delta_{x_n}) = P(\delta_x) = p(x)$. So, p is continuous.

(B) Consider $\nu \in \Delta(X)$ with finite support. By definition, $x^*(m_\nu) = \int_X \nu(dx)\, x^*(x) = \sum_{x \in \operatorname{supp} \nu} \nu(\{x\}) x^*(x) = x^*(\sum_{x \in \operatorname{supp} \nu} \nu(\{x\}) x)$ for every $x^* \in X^*$. As X^* is total, we have $m_\nu = \sum_{x \in \operatorname{supp} \nu} \nu(\{x\}) x$, and therefore $\nu \in \Delta(X)_0$. If $\sum_{x \in \operatorname{supp} \nu} \nu(\{x\}) P(\delta_x) < P(\nu)$, then $\theta = \sum_{x \in \operatorname{supp} \nu} \nu(\{x\}) 1_{\delta_x} - 1_\nu$ contradicts condition (1) as

$$\sum_{\mu \in \operatorname{supp} \theta} \theta(\mu) P(\mu) = \sum_{x \in \operatorname{supp} \nu} \nu(\{x\}) P(\delta_x) - P(\nu) < 0$$

and

$$\sum_{\mu \in \operatorname{supp} \theta} \theta(\mu) m_\mu = \sum_{x \in \operatorname{supp} \nu} \nu(\{x\}) x - m_\nu = 0$$

We can similarly rule out $P(\nu) < \sum_{x \in \operatorname{supp} \nu} \nu(\{x\}) P(\delta_x)$. So,

$$P(\nu) = \sum_{x \in \operatorname{supp} \nu} \nu(\{x\}) P(\delta_x) = \int_X \nu(dx)\, p(x) \tag{5}$$

Consider $\mu \in \Delta(X)_O$. By Lemma 2.1(F), $\mu \in \Delta(X)_0$. Lemma 2.1(E) implies $T^{-1}(\mu) \in \Delta(O)_0$. As O is compact metric, it is separable. So, there is a sequence $(\mu_n) \subset \Delta(O)$ converging to $T^{-1}(\mu)$ such that $|\operatorname{supp} \mu_n| < \infty$ for every $n \in \mathcal{N}$ (Parthasarathy [7], Chapter II, Theorem 6.3). It follows that $m_{\mu_n} = \sum_{x \in \operatorname{supp} \mu_n} \mu_n(\{x\}) x \in O$ for every $n \in \mathcal{N}$. Therefore, $(\mu_n) \subset \Delta(O)_0$. By Lemma 2.1(D), $(T(\mu_n)) \subset \Delta(X)_0 \cap \Delta(X)_O$. Using (5), we have $P(T(\mu_n)) = \int_X T(\mu_n)(dx)\, p(x) = \int_O \mu_n(dx)\, p(x)$ for every $n \in \mathcal{N}$. Using Lemma 2.1(B), $\lim_n T(\mu_n) = T(\lim_n \mu_n) = T \circ T^{-1}(\mu) = \mu$. By Lemma 2.1(F), $m_\mu \in O$. Using (A), as O is compact, the restriction of p to O is in $\mathcal{C}(O)$. Hence, $P(\mu) = P(\lim_n T(\mu_n)) = \lim_n P(T(\mu_n)) = \lim_n \int_O \mu_n(dx)\, p(x) = \int_O T^{-1}(\mu)(dx)\, p(x) = \int_X \mu(dx)\, p(x) = p(m_\mu)$, as required. ■

Proof of Lemma 3.6 Suppose u is more risk averse than v and there exist $y, z \in O$ such that, either

(a) $u(z) \geq u(y)$ and $v(z) < v(y)$, or

(b) $u(z) < u(y)$ and $v(z) \geq v(y)$.

If (a), then $\delta_z \in A(y,u) \setminus A(y,v)$. If (b) and $v(z) > v(y)$, then $\delta_y \in A(z,u) \setminus A(z,v)$. In both cases, u is not more risk averse than v, which is a contradiction.

Suppose (b) and $v(z) = v(y)$. As v is non-constant, there exists $x \in O$ such that, either $v(x) > v(y)$ or $v(x) < v(y)$.

Suppose $v(x) > v(y)$. As O is convex, $c(t) = tx + (1-t)z \in O$ for every $t \in (0,1)$. As u is continuous and v is concave, there exists $t \in (0,1)$, sufficiently close to 0, such that $u(c(t)) < u(y)$ and $v(c(t)) \geq tv(x) + (1-t)v(z) > v(y)$. Thus, $\delta_y \in A(c(t),u) \setminus A(c(t),v)$, which contradicts the hypothesis that u is more risk averse than v.

Suppose $v(x) < v(y)$. Let $\mu(t) = t\delta_x + (1-t)\delta_y \in \Delta(O)_0$. Then, $U(\mu(t),u) = tu(x) + (1-t)u(y) > u(z)$ for some $t \in (0,1)$ sufficiently close to 0. As $U(\mu(t),v) = tv(x) + (1-t)v(y) < v(z)$, we have $\mu(t) \in A(z,u) \setminus A(z,v)$, which contradicts the hypothesis that u is more risk averse than v. ■

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