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A dual formulation of bidding behaviour in sealed bid auctions

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Abstract

We formalise 'bidding behaviour' as a bidder's choice of the 'mean winning probability' at the interim stage of first-price and second-price sealed bid auctions. This formulation simplifies and sharpens the analysis of bidding behaviour by virtue of confining it to the unit interval. As an application, we show that the optimal mean winning probability increases if and only if the bidder's valuation of the prize increases. Our formulation of bidding behaviour is rationalised by duality results showing that optimal mean winning probabilities correspond to optimal bid distributions.

JEL classification: D44

Key words: first-price auction, second-price auction, mean winning probability, bidding behaviour, duality

1 Introduction

Consider first-price and second-price sealed bid auctions of a prize. At the interim stage of the Bayesian game model of each of these auctions, if B is Bidder 1's valuation of the prize and G is the distribution function of the highest bid among the rivals (henceforth, the 'rival distribution' function), then Bidder 1 has to choose a bid distribution to maximise her expected net payoff.[1] If the realised bids make her the winner, then her net payoff is B minus the payment determined by the auction form and the realised bids; otherwise, she gets payoff zero. Bidder 1's decision problem raises a natural question: *given G, how does a variation of B affect her bidding behaviour?*

Lebrun [5] addresses the question: in a first-price auction with independent private values, how does a variation of some bidder's prior valuation distribution affect the equilibrium bid functions?[2] Although it bears

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[1] Equations (10), (11), and (18)-(21) express G, B, and Bidder 1's expected net payoff, respectively, in terms of the game generated by an auction form.

[2] Similar questions are studied in Hopkins and Kornienko [4].

a passing resemblance to our question insofar as it concerns a link between valuations and bidding in auctions, this question is fundamentally different because it concerns a Bayesian game's *ex ante* stage and the game's equilibrium mapping, while our question concerns the game's interim stage and a particular player's best response mapping. Clearly, the dependent variables in these questions are different and so are the parameters whose variational effects are sought to be characterised.

Returning to our question, we observe that, given the rival distribution, Bidder 1's bids are ultimately a means of determining her expected probability of winning along with the entailed expected cost. Accordingly, instead of analysing Bidder 1's behaviour at the interim stage directly in terms of bid distributions, we shall do so indirectly in terms of Bidder 1's 'mean winning probability', which is after all the ultimate object of interest for her. Unlike bid distributions that are typically only partially ordered, mean winning probabilities are real numbers that are completely ordered. This feature simplifies and sharpens the analysis of bidding behaviour.

The first step towards the indirect formulation is the intermediate formulation of Bidder 1's problem wherein distributions over winning probabilities replace bid distributions as Bidder 1's choice variable. This shift is plausible and possible because a distribution over bids (resp., the unit interval) generates a dual distribution over the unit interval (resp., bids) *via* the rival distribution function G (resp., the quantile function dual to G).

The indirect formulation involves a further shift from the intermediate formulation by replacing distributions over the unit interval with their means as Bidder 1's choice variable. For a class of rival distribution functions, Bidder 1's problem in the indirect formulation is tractable in the σ-additive setting, i.e., when the mean winning probabilities are generated by σ-additive distributions over the unit interval. The problem is more generally tractable in the finitely additive setting, i.e., when the mean winning probabilities are generated by finitely additive distributions over the unit interval. The nature of distributions underlying the mean winning probabilities is significant, not so much for generating the probabilities *per se*, but for determining Bidder 1's costs of implementing all the mean winning probabilities and thereby shaping her incentives for choosing a particular mean winning probability; see Section 1.1 for further elucidation of this point.

We show in Sections 3 and 4 that the indirect formulation yields an elementary, sharp, and general answer to our question for both auction forms: *a higher optimal mean winning probability (i.e., 'more aggressive bidding behaviour') corresponds to a higher* B. For both auction forms, Bidder 1's choice space in the indirect formulation is just the unit interval, the objective function on the unit interval is sufficiently regular that the characterisation of a globally optimal choice involves the most rudimentary marginal considerations, and the variational result yields a complete ordering of optimal choices conditional on B.

Since the direct formulation is the standard model in the auction context, the indirect and intermediate formulations need to be rationalised in terms of the direct formulation. In order to do so, we shall demonstrate in Section 5 that the solutions – contingent on B and G – of all three formulations of Bidder 1's problem are bijectively related. These dualities hold for both auction forms, provided Bidder 1's decision problems are tractable when choosing from σ-additive bid distributions in the direct formulation, choosing from σ-additive distributions over the unit interval in the intermediate formulation, and choosing from the means of σ-additive distributions over the unit interval in the indirect formulation.

For both auction forms, the duality between the intermediate and the indirect formulations holds even more generally if Bidder 1 chooses from finitely additive distributions over the unit interval in the intermediate formulation and chooses from mean winning probabilities that are generated by finitely additive distributions over the unit interval in the indirect formulation.

These dualities also demonstrate the indirect formulation's canonicity inasmuch as every problem of optimally randomising over bids from some bounded interval is shown to be equivalent to the problem of choosing a probability from the unit interval, provided the settings of the various bidding problems meet some regularity conditions.

In the rest of the paper, Section 2 states some formal preliminaries, Section 6 summarises this paper's findings, and Appendix contains proofs of the lemmata. While these sections do not require prefatory remarks, it may be useful to outline Sections 3-5 in somewhat greater detail prior to the formal analysis.

1.1 Bidder 1's problem

Section 3 is devoted to the construction of the indirect formulation. Given G and the auction form, Bidder 1's bid determines her probability of winning the prize and the resulting expected cost conditional on the bid. Therefore, a bid distribution generates a distribution over winning probabilities, the mean winning probability, and the resulting expected cost. This suggests an indirect formulation of Bidder 1's problem wherein she chooses the mean winning probability c that is to be generated by a bid distribution at the least possible cost.[3] Given this set-up, we identify the chosen c with 'bidding behaviour'. As Bidder 1's expected benefit from choosing c is cB, the critical issue is whether the minimum expected costs of generating various values of c allow a characterisation of the optimal c that yields a predictable qualitative

[3] Such dual modelling of decision problems is commonplace in economics. For instance, a firm may represent technology by a production function and optimally choose inputs to indirectly determine the output, or equivalently represent technology by the cost function (with efficient input choices embedded in it) and directly choose the optimal output.

link between B and the optimal c. This problem is set up with increasing generality as follows.

If Bidder 1 wishes to generate a winning probability c, then the least expected cost of doing so by means of a single deterministic bid is $g(c)$, as defined by Equation (4). The restriction to deterministic bids results in Bidder 1's problem being: choose c to maximise $cB - g(c)$. By Lemma 3.4, this deterministic version of the indirect model is tractable if g is convex.

However, as g may not be convex, the problem is generalised as follows. For every c, let $e(c)$ be the minimised expectation of g on the set of σ-additive distributions over the unit interval with mean c. In effect, c is reinterpreted as the mean winning probability instead of being the winning probability. This randomisation does not change Bidder 1's expected benefit from winning as, for every c, the expected valuation derived from a distribution with mean c is simply cB. Thus, Bidder 1's problem with the reinterpreted c becomes: choose c to maximise $cB - e(c)$.[4] Lemma 3.5 implies that this σ-additive version of the indirect model is tractable if g is continuous. Lemma 3.3 characterises a continuous g for the two auction forms in terms of the rival distribution function G.

However, g is generally neither convex nor continuous. This lacuna is filled by allowing Bidder 1 to implement mean winning probabilities using finitely additive distributions over the unit interval. For every c, let $\eta(c)$ be the minimised expectation of g on the set of finitely additive distributions over the unit interval with mean c. Then, Bidder 1's problem becomes: choose c to maximise $cB - \eta(c)$. Lemma 3.6 shows that this problem is tractable even if g is discontinuous and non-convex.

1.2 Solving Bidder 1's problem

In Section 4, we analyse Bidder 1's problem. Lemma 4.1 characterises optimal mean winning probabilities in terms of B, which is nothing but an expression of the canonical rule to equalise benefit and cost at the margin. Using this characterisation, Theorem 4.3 shows that the optimal winning probability increases if and only if B does so.

1.3 Duality results

Section 5 relates different models of Bidder 1's problem to each other. Three specifications determine each model: (1) whether the first-price or the second-price auction is being considered, (2) whether the formulation is direct, intermediate, or indirect, and (3) whether the setting involves Bidder 1 choosing σ-additive or finitely additive distributions.

Theorems 5.1, 5.2, and Remark 5.3 assert that the indirect and intermediate models are equivalent for both auction forms and both settings in

[4] See the discussion following (13) for a formal justification of this objective function.

the sense that the solvability of either model implies the solvability of the other model and these solutions are bijectively related. These dualities are broad in scope and hold generally without supplementary hypotheses. The canonicity of the indirect formulation is demonstrated in Remark 5.4.

The other duality results are shown for the σ-additive setting only. Given a continuous rival distribution function, Theorem 5.6 shows that the mapping of distributions over the unit interval to bid distributions using the quantile function is a bijection. Theorem 5.8 shows that this bijection preserves expected costs and mean winning probabilities across the intermediate and direct formulations of bidding behaviour for both auction forms. Hence, for both auction forms, the bijection preserves Bidder 1's preference ordering when it maps distributions over winning probabilities to bid distributions (Corollary 5.9). Finally, for both auction forms, Bidder 1's optimal choices in the direct and intermediate models are a dual pair with respect to the aforementioned bijection (Corollary 5.10). Combining this result with Theorem 5.2 and Remark 5.3 yields a duality between the direct and the indirect formulations.

2 Formal preliminaries

The following definitions and conventions will apply throughout this paper.

We say that a set is countable if it is either finite or denumerable. A set X is said to be a co-countable subset of a set Y if $X \subset Y$ and $Y \setminus X$ is countable. Consider a real-valued function f defined on a domain $\mathrm{Dom}\,(f)$. Suppose $\mathrm{Dom}\,(f) \subset \Re$; throughout, $\subset$ is to be understood in its weak form. If f is differentiable at $x \in \mathrm{Dom}\,(f)$, then $Df(x)$ denotes the derivative of f at x. f is said to be increasing (resp., strictly increasing) if $x, y \in \mathrm{Dom}\,(f)$ and $x > y$ implies $f(x) \geq f(y)$ (resp., $f(x) > f(y)$). If f is increasing and $(x - \epsilon, x) \subset \mathrm{Dom}\,(f)$ (resp., $(x, x + \epsilon) \subset \mathrm{Dom}\,(f)$) for some $\epsilon > 0$, then $f(x-) = \lim_n f(x - 1/n) = \sup f((x - \epsilon, x))$ (resp., $f(x+) = \lim_n f(x + 1/n) = \inf f((x, x + \epsilon))$). If $\mathrm{Dom}\,(f) = \Re$ and f is a distribution function, then we assume f is right-continuous and set $\Phi(f) = \{c \in [0, 1] \mid |f^{-1}(\{c\})| > 1\}$; so, $c \in \Phi(f)$ corresponds to a flat portion of f's graph. If $\mathrm{Dom}\,(f) \subset \Re^n$, then f is said to be strictly increasing if $x, y \in \mathrm{Dom}\,(f)$, $x \leq y$, and $x \neq y$ implies $f(x) < f(y)$.

For an interval $I \subset \Re$, $\mathcal{B}(I)$ is the Borel σ-algebra on I and $\Delta(I)$ is the set of σ-additive distributions on $(I, \mathcal{B}(I))$. For $E \in 2^{\Re}$, we say $E \in \mathcal{L}$ if $\mathrm{Leb}^*(T) = \mathrm{Leb}^*(T \cap E) + \mathrm{Leb}^*(T \setminus E)$ for every $T \in 2^{\Re}$, where Leb^* is the Lebesgue outer measure on $2^{\Re}$. $E \in \mathcal{L}$ is called a Lebesgue measurable set. Using Caratheodory's theorem (Bruckner et al. [2], Theorem 2.32), $\mathcal{L}$ is a σ-algebra and the restriction of Leb^* to $\mathcal{L}$ is called the Lebesgue measure, which is denoted by Leb. All references to absolute continuity of measures shall be with respect to Leb.

3 The cost function

In this section, we derive Bidder 1's expected cost of efficiently ensuring a mean winning probability c in three increasingly general settings: with a deterministic choice in Section 3.1, with a σ-additive distribution in Section 3.2, and with a finitely additive distribution in Section 3.3.

Consider either the first-price or second-price sealed bid auction of a prize with the set of bidders $\{1, \ldots, n\}$, with $n \geq 2$. For $i \in \{2, \ldots, n\}$, let y_i be the nonnegative-valued random variable that generates bidder i's bid.

Throughout this paper, let $C = [0, 1]$, let G be the distribution function of the random variable $y^* = \max\{y_2, \ldots, y_n\}$ such that

$$0 = G(0) < G(b) \text{ for } b > 0, \quad \inf G^{-1}(\{1\}) = \beta_1 < \infty, \quad G(\beta_1-) = 1 \tag{1}$$

and let $b : C \to [0, \beta_1]$ be the corresponding quantile function defined by

$$b(c) = \inf[G^{-1}([c, 1]) \cap \Re_+] \tag{2}$$

Additional assumptions about G and b will be stated when required; see Equation (10) for G's derivation from the game generated by an auction form.

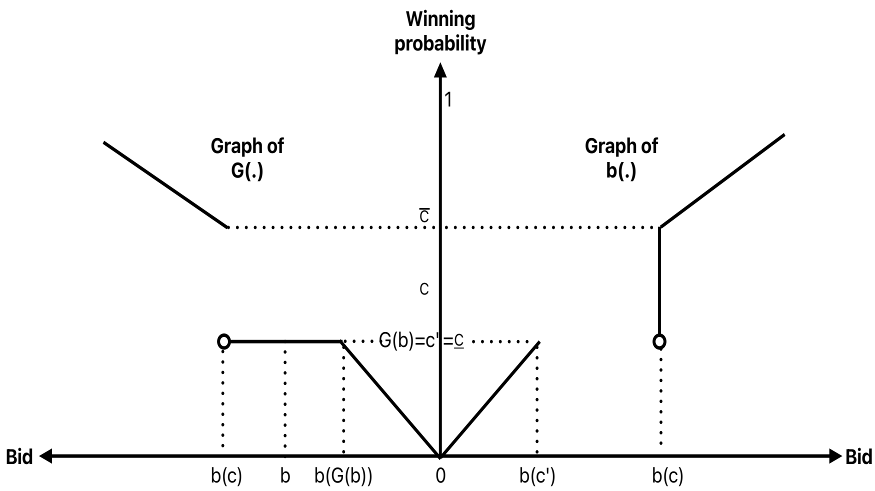

Figure 1: Duality between G and b

For both auction forms, we assume that Bidder 1 wins whenever she is a highest bidder. Therefore, if Bidder 1 bids b, then she wins with probability $G(b)$. Figure 1 illustrates the duality between the rival distribution function G and the corresponding quantile function b. In this diagram, the negative part of the horizontal axis is G's domain, the interval $C = [0, 1]$ in the vertical axis is G's codomain and b's domain, and the positive part of the horizontal axis is b's codomain. Evidently, $b(0) = 0$, $b(1) = \beta_1$, $b(c) = \inf G^{-1}([c, 1]) > 0$ for $c \in (0, 1]$, and jumps (resp., flat portions) in G's graph correspond to flat portions (resp., jumps) in b's graph. If $c \in C$, then $G^{-1}([c, 1]) \cap \Re_+$ is bounded below by 0, and as G is right-continuous, $G^{-1}([c, 1])$ is closed in $\Re$. So, $b(c) \in G^{-1}([c, 1])$, i.e., $G \circ b(c) \geq c$. If $b < b(c)$, then $b \notin G^{-1}([c, 1])$, i.e., $G(b) < c$. *Thus, $b(c)$ is the minimum bid by Bidder 1 that yields a winning probability greater than or equal to c.* Let

$$\underline{c} = \inf b^{-1}(\{b(c)\}) \quad \text{and} \quad \bar{c} = \sup b^{-1}(\{b(c)\}) \tag{3}$$

for $c \in C$. It follows that $\underline{c}, \bar{c} \in C$, $\bar{0} = 0$, and $\bar{c} \geq c > 0$ for $c > 0$. We collect a number of useful facts about G and b in the following lemma.

Lemma 3.1 *Given G,*

(A) b is nonnegative, bounded, increasing, measurable, and left-continuous,

(B) b (resp. G) is continuous if and only if G (resp., b) is strictly increasing on $[0, \beta_1]$ (resp., C),

(C) $b(\bar{c}) = b(c)$ for every $c \in C$,

(D) $\bar{c} = G \circ b(\bar{c}) \in G([0, \beta_1])$ for every $c \in C$,

(E) $c < G \circ b(c)$ for $c < \bar{c}$, and

(F) $G([0, \beta_1]) = \{c \in C \mid c = \bar{c}\}$. So, $G \circ b \circ G(\beta) = G(\beta)$ for every $\beta \in [0, \beta_1]$.

(G) If G is continuous, then $b^{-1}([0, x]) = G([0, x])$ for every $x \in [0, \beta_1]$.

(H) If G is continuous and strictly increasing, then $G(B) \in \mathcal{B}(C)$ for every $B \in \mathcal{B}([0, \beta_1])$.

3.1 Cost with deterministic choice

If Bidder 1 wins, then she pays her own bid in a first-price auction and y^* in a second-price auction; otherwise, she pays nothing. For $c \in C$, if Bidder 1 bids $b(c)$, then the implied expected payment by Bidder 1 is

$$g_1(c) = b(c) G \circ b(c) \quad \text{and} \quad g_2(c) = \int_{[0, b(c)]} y \, dG(y) \tag{4}$$

in the first-price auction and second-price auction respectively. Note that

$$g_2(c) = g_1(c) - \int_{[0, b(c)]} dy \, G(y) \tag{5}$$

Evidently, $g_2 < g_1$ over $(0, 1]$, i.e., the expected payment entailed by a deterministically implemented positive winning probability is lower in a first-price auction than in a second-price auction.

Henceforth, a statement concerning g is to be understood as holding for both g_1 and g_2, unless g_1 or g_2 is specified explicitly.

Lemma 3.2 *The expected payment function $g : C \to \Re$ given by Equation (4) is nonnegative with $g(0) = 0$, bounded, increasing, and integrable.*

The next result characterises the continuity of g_1 and g_2 in terms of G's properties.

Lemma 3.3 *Consider G, b, g_1, and g_2 given by Equations (1), (2), and (4).*

(A) If G is strictly increasing on $[0, \beta_1]$ and continuous, then $G : [0, \beta_1] \to C$ is a homeomorphism with function inverse $b : C \to [0, \beta_1]$).

(B) G is strictly increasing on $[0, \beta_1]$ and continuous if and only if g_1 is continuous.

(C) G is continuous if and only if g_2 is continuous.

Simple examples show that the discontinuities of g_1 and g_2 can occur on $(0, 1)$. Hence, neither g_1 nor g_2 is necessarily convex. However, if g is convex, then it renders Bidder 1's problem tractable without resorting to randomisation.

Lemma 3.4 *Consider g given by Equation (4). If g is convex, then*

(A) $g(c) \leq \int_C \lambda(dx)\, g(x)$ for $c \in C$ and $\lambda \in \Delta(C)$ such that $\int_C \lambda(dx)\, x = c$,

(B) g is nonnegative, bounded, convex, increasing, $e(0) = 0$,

(C) g is continuous on $[0, 1)$,

(D) g is subdifferentiable on $(0, 1)$, differentiable on a co-countable subset of $(0, 1)$, twice differentiable Leb-*a.e., and*

(E) for some $c_0 \in [0, 1]$, g is constant over $[0, c_0]$ and strictly increasing over $[c_0, 1]$.

On the other hand, if g is not convex, then Bidder 1 may lower her expected payment for implementing a winning probability $c \in C$ by randomising over C using a distribution with mean c. This randomisation does not affect Bidder 1's expected gross payoff cB as it is linear in c.

3.2 Cost with σ-additive distributions

Suppose the buyer chooses a distribution $\lambda \in \Delta(C)$ to randomise over C. The resulting mean winning probability is $m(\lambda) = \int_C \lambda(dc)\, c$ for both auction forms, the implied bid distribution is $\lambda \circ b^{-1} \in \Delta([0, \beta_1])$, and the expected payment is $L(g_1, \lambda) = \int_C \lambda(dx)\, g_1(x)$ for the first-price auction and $L(g_2, \lambda) = \int_C \lambda(dx)\, g_2(x)$ for the second-price auction. As the set of implementations of the mean winning probability c is $\Delta(C, c) = \{\lambda \in \Delta(C) \mid m(\lambda) = c\}$, the least expected cost of implementing c using σ-additive distributions is

$$e_1(c) = \inf L(g_1, \Delta(C, c)) \quad \text{and} \quad e_2(c) = \inf L(g_2, \Delta(C, c)) \tag{6}$$

for the first-price auction and the second-price auction respectively. *Henceforth, a statement concerning e is to be understood as holding for both e_1 and e_2, unless e_1 or e_2 is specified explicitly.*

Lemma 3.4(A) amounts to saying that, if g is convex, then $e(c) = g(c) = L(g, \delta_c)$ for every $c \in C$ and e has various regularity properties. The following result provides another sufficient condition for the infimum in Equation (6) to be achieved on the set $\Delta(C, c)$ and for e to have essentially the same regularity properties.

Lemma 3.5 *Consider g and e given by Equations (4) and (6) respectively. If g is continuous, then*

(A) for every $c \in C$, $e(c) = L(g, \lambda_c) \in \Re$ for some $\lambda_c \in \Delta(C, c)$,

(B) e is nonnegative, bounded, and $e(0) = 0$,

(C) e is convex,

(D) e is increasing,

(E) e is continuous,

(F) e is subdifferentiable on $(0, 1)$, differentiable on a co-countable subset of $(0, 1)$, twice differentiable Leb-*a.e., and*

(G) for some $c_0 \in [0, 1]$, e is constant over $[0, c_0]$ and strictly increasing over $[c_0, 1]$.

3.3 Cost with additive distributions

Since g may be non-convex and discontinuous, there remains the problem of deriving the efficient expected payment when g is assumed to have only the general properties noted in Lemma 3.2. In order to derive the efficient expected payment implied by a general g, we expand the set of available randomisations from σ-additive to all finitely additive distributions over C.

Let $B(C)$ be the set of uniform limits of finite linear combinations of the indicator functions of sets in $\mathcal{B}(C)$.[5] Given the supremum norm, $B(C)$ is a Banach space. As the identity function on C and g are bounded and Borel measurable, both functions are in $B(C)$. Let $\mathrm{ba}(C)$ be the set of bounded, finitely additive real-valued functions defined on $\mathcal{B}(C)$.[6] Given the variation norm, $\mathrm{ba}(C)$ is a Banach space that is isomorphic to the continuous dual of $B(C)$ (Dunford and Schwartz [3], Theorem IV.5.1). Denote the set of finitely additive probability measures on $(C, \mathcal{B}(C))$ by $P(C) \subset \mathrm{ba}(C)$.

Given $\lambda \in P(C)$, since the identity function on C and g are in $B(C)$, the Dunford-Schwartz integrals (Dunford and Schwartz [3], Chapter III) $m(\lambda) = \int_C \lambda(dx)\, x$ and $L(g, \lambda) = \int_C \lambda(dx)\, g(x) \geq 0$ are well-defined.

Given $P(C, c) = \{\lambda \in P(C) \mid m(\lambda) = c\}$, the least expected cost of implementing c using finitely additive distributions is

$$\eta_1(c) = \inf L(g_1, P(C, c)) \quad \text{and} \quad \eta_2(c) = \inf L(g_2, P(C, c)) \tag{7}$$

for the first-price auction and the second-price auction respectively. As $\Delta(C) \subset P(C)$, we have $\Delta(C, c) \subset P(C, c)$. *Henceforth, a statement concerning η is to be understood as holding for both η_1 and η_2, unless η_1 or η_2 is specified explicitly.*

Lemma 3.6 *If g and η are given by Equations (4) and (7), respectively, then*

(A) for every $c \in C$, $\eta(c) = L(g, \lambda) \in \Re$ for some $\lambda \in P(C, c)$,

(B) η is nonnegative, bounded, and $\eta(0) = 0$,

(C) η is convex on C,

(D) η is increasing,

(E) η is continuous on $[0, 1)$,

(F) η is subdifferentiable on $(0, 1)$, differentiable on a co-countable subset of $(0, 1)$, twice differentiable Leb-*a.e., and*

(G) for some $c_0 \in [0, 1]$, η is constant over $[0, c_0]$ and strictly increasing over $[c_0, 1]$.

Part (G) implies that, for $c \in [c_0, 1]$, the expected payment $\eta(c)$ ensures that the mean winning probability is exactly c.

[5] $B(C)$ includes all bounded Borel measurable real-valued functions on C, and therefore all continuous real-valued functions on C.

[6] $\mathrm{ba}(C)$ includes all Borel measures on $(C, \mathcal{B}(C))$

4 The optimal choice

Let B denote Bidder 1's valuation of the prize. If her mean winning probability is $c \in C$, then her expected valuation is cB. Therefore, the expected payoff from implementing the mean winning probability c efficiently is $cB - \eta(c)$ (resp., $cB - e(c)$) if c is implemented using finitely additive distributions from $P(C)$ (resp., σ-additive distributions from $\Delta(C)$). In this section, we shall analyse Bidder 1's problem of choosing c to maximise

$$V(B, c) = cB - \eta(c) \tag{8}$$

This setting admits a discontinuous g and therefore a discontinuous G.

Lemma 4.1 *Suppose $B \in \Re_+$, and g, η, and V are given by Equations (4), (7), and (8) respectively.*

(A) $1 \in \arg\max_C V(B, .)$ *if and only if* $B \geq [\eta(1) - \eta(c)]/(1 - c)$ *for every* $c \in [0, 1)$.

(B) $0 \in \arg\max_C V(B, .)$ *if and only if* $B \leq \eta(c)/c$ *for every* $c \in (0, 1]$.

(C) If $c_0 > 0$, *then* $[0, c_0) \cap \arg\max_C V(B, .) = \emptyset$. *If η is continuous at 1, then* $\emptyset \neq \arg\max_C V(B, .) \subset [c_0, 1]$. *If η is discontinuous at 1, then* $1 \notin \arg\max_C V(B, .)$.

(D) $\arg\max_C V(B, .)$ *is a convex set that may be empty.*

(E) If $c \in (0, 1)$, *then* $c \in \arg\max_C V(B, .)$ *if and only if* $B \in \partial\eta(c)$. *If η is differentiable at* $c \in (0, 1)$, *then* $c \in \arg\max_C V(B, .)$ *if and only if* $B = D\eta(c)$.

(F) $\{B \in \Re_+ \mid c \in \arg\max_C V(B, .)\}$ *is convex for every $c \in C$ and is a singleton for every c in a co-countable subset of* $(0, 1)$.

Remark 4.2 *Suppose the above problem is modified so that g given by Equation (4) is continuous, e is given by Equation (6), and*

$$V(B, c) = cB - e(c) \tag{9}$$

Then, parts (A), (B), (E), and (F) of Lemma 4.1 hold with η replaced by e, while parts (C) and (D) are strengthened to state that $\arg\max_C V(B, .)$ *is a nonempty and convex subset of* $[c_0, 1]$.

In special circumstances, the winning probabilities chosen in the two auction forms can be compared. Suppose G is strictly increasing and continuously differentiable. Then, by Lemma 3.1, b is determined by the identity $G \circ b(c) = c$, b is continuously differentiable, and it follows from Equation (5) that $Dg_2 \leq Dg_1$. Suppose g_1 and g_2 are convex; let c_1 and c_2 be the optimal winning probabilities in the first-price auction and the second-price auction respectively; and suppose g_1 and g_2 are differentiable at c_1 and c_2 respectively. By Lemma 3.4, $g_1 = e_1$ and $g_2 = e_2$, and by Remark 4.2, $Dg_2(c_1) \leq Dg_1(c_1) = B = Dg_2(c_2)$, which implies $c_1 \leq c_2$. So, *if g_1 and g_2 are convex, then it is optimal to choose winning probabilities deterministically, and for a given valuation, the optimal winning probability in the second-price auction exceeds the optimal winning probability in the first-price auction.*

Finally, we investigate the variational relationship between the optimal choice c and the valuation B. The key results are parts (C) and (E).

Theorem 4.3 *Consider g, η, and V defined by Equations (4), (7), and (8) respectively. Suppose $B_v \geq 0$, $B_u \geq 0$, $c_v \in \arg\max_C V(B_v, .)$, and $c_u \in \arg\max_C V(B_u, .)$.*

(A) If $B_v = B_u$, then $\arg\max_C V(B_v, .) = \arg\max_C V(B_u, .)$.

(B) If $B_v \geq B_u$ and $c_u = 1$, then $c_v = 1$.

(C) If $B_v > B_u$, then $c_v \geq c_u$.

(D) If $B_v > B_u$ and e is differentiable at c_u and c_v, then $c_v > c_u$.

(E) If $c_v > c_u$, then $B_v \geq B_u$.

Proof. Consider g, e, V, B_v, B_u, c_v, and c_u *ex hypothesi.*

(A) is obvious.

(B) Using Lemma 4.1(A), if $c_u = 1$, then $B_v \geq B_u \geq [\eta(1) - \eta(c)]/(1 - c)$ for every $c \in [0, 1)$. Again using Lemma 4.1(A), $c_v = 1$.

(C) Suppose $B_v > B_u$. If $c_v = 1$, the conclusion holds trivially. If $c_u = 1$, then $c_v = 1$ by (B) and the conclusion holds.

Suppose $c_u, c_v \in (0, 1)$. By Lemma 4.1(E), $B_u \in \partial\eta(c_u)$ and $B_v \in \partial\eta(c_v)$. Thus, $\eta(c_v) - \eta(c_u) \geq B_u(c_v - c_u)$ and $\eta(c_u) - \eta(c_v) \geq B_v(c_u - c_v)$. It follows that $(B_u - B_v)(c_v - c_u) \leq 0$. As $B_v > B_u$, we have $c_v \geq c_u$.

(D) By (C), $c_v \geq c_u$. As e is differentiable at c_u and c_v, Lemma 4.1(E) implies $B_v - D\eta(c_v) = 0 = B_u - D\eta(c_u)$. As $B_v > B_u$, we have $D\eta(c_v) > D\eta(c_u)$. Hence, $c_v \neq c_u$. So, $c_v > c_u$.

(E) Consider $c_u, c_v \in C$ such that $c_u < c_v$.

Suppose $c_u = 0$. Then, $c_v B_u - \eta(c_v) = V(B_u, c_v) \leq V(B_u, c_u) = 0 = V(B_v, 0) \leq V(B_v, c_v) = c_v B_v - \eta(c_v)$. Thus, $c_v(B_v - B_u) \geq 0$. As $c_v > 0$, we have $B_v \geq B_u$.

Suppose $c_v = 1$. As $c_u < c_v = 1$, Lemma 4.1(A) implies $B_v \geq [\eta(1) - \eta(c_u)]/(1 - c_u)$. If $c_u = 0$, then Lemma 4.1(B) implies $B_v \geq \eta(1)/1 \geq B_u$. If $c_u > 0$, then $c_u \in (0, 1)$. Therefore, Lemma 4.1(E) implies $B_u \in \partial\eta(c_u)$. Hence, $B_v \geq [\eta(1) - \eta(c_u)]/(1 - c_u) \geq B_u$.

Finally, suppose $c_u, c_v \in (0, 1)$. By Lemma 4.1(E), $B_u \in \partial\eta(c_u)$ and $B_v \in \partial\eta(c_v)$. Thus, $\eta(c_v) - \eta(c_u) \geq B_u(c_v - c_u)$ and $\eta(c_u) - \eta(c_v) \geq B_v(c_u - c_v)$. It follows that $(B_u - B_v)(c_v - c_u) \leq 0$. As $c_v > c_u$, we have $B_v \geq B_u$. ■

Remark 4.4 *Suppose the problem addressed in Theorem 4.3 is modified so that g given by Equation (4) is continuous, e is given by Equation (6), and V is given by Equation (9). Then, it is easy to verify that the results in Theorem 4.3 continue to hold.*

Theorem 4.3 relates the ordering of c_v and c_u to the ordering of B_v and B_u. Equation (11) derives B_v and B_u from distribution $\mu \in \Delta(T)$ and utilities v and u respectively. The following result provides conditions on the exogenous data v, u, and μ that predict the ordering of B_v and B_u.

Theorem 4.5 *Suppose $T = \Re^n_+$ for some $n \in \mathcal{N}$ and $\mu \in \Delta(T)$. Consider continuous functions $v : T \to \Re$ and $u : T \to \Re$ such that $B_v = \int_T \mu(dt)\, v(t) < \infty$ and $B_u = \int_T \mu(dt)\, u(t) < \infty$.*

(A) If $v \geq u$ on T, then $B_v \geq B_u$.

Suppose $u(0) = v(0) = 0$, $\neg v \geq u$ on T, and $\neg u \geq v$ on T.

(B) Then, there exists $\tau \in T \setminus \{0\}$ such that $v(\tau) - u(\tau) = 0$.

(C) Suppose u and v are risk averse and strictly increasing. If u is more risk averse than v and $B_v \geq v(\tau)$, then $B_v \geq B_u$.

Proof. Consider μ, u, v, B_u, and B_v *ex hypothesi.*

(A) follows immediately from the definitions.

(B) As $\neg v \geq u$ and $\neg u \geq v$, there exist $a, b \in T \setminus \{0\}$ such that $v(a) - u(a) < 0$ and $v(b) - u(b) > 0$ respectively. As v and u are continuous, so is $v - u$. Hence, there is some convex combination of a and b, say $\tau \in T \setminus \{0\}$, such that $v(\tau) - u(\tau) = 0$.

(C) As u is more risk averse than v, there is an increasing concave function $f : v(T) \to \Re$ such that $u = f \circ v$. It follows that $f(0) = 0$ and $y \mapsto f(y)/y$ is decreasing over $v(T)$. As $\tau \in T \setminus \{0\}$ and v is strictly increasing, $f \circ v(\tau) = u(\tau) = v(\tau) > 0$.

Since $B_v \geq v(\tau)$, we have $f(B_v)/B_v \leq f \circ v(\tau)/v(\tau) = 1$. So, $B_u = B_{f \circ v} = \int_T \mu(dy)\, f \circ v(y) = \int_\Re \mu \circ v^{-1}(dy)\, f(y) \leq f(\int_\Re \mu \circ v^{-1}(dy)\, y) = f(\int_T \mu(dy)\, v(y)) = f(B_v) \leq B_v$. ■

Given u, v, and τ, satisfaction of the condition $B_v \geq v(\tau)$ depends on $\mu \circ v^{-1}([v(\tau), \infty))$. For instance, consider $\mu = \alpha\delta_{\tau'} + (1 - \alpha)\delta_0$, where $\alpha \in (0, 1)$ and $\tau' \in T$ such that $\tau' \geq \tau$. Then, $B_v = \alpha v(\tau') + (1 - \alpha)v(0) = \alpha v(\tau')$. If $\mu \circ v^{-1}([v(\tau), \infty)) = \mu(\{\tau'\}) = \alpha \geq v(\tau)/v(\tau')$, then $B_v \geq v(\tau)$.

5 Duality

We shall consider three models of Bidder 1's problem: the indirect, the intermediate, and the direct model. The analysis thus far has concerned the first model. In this section, we show dualities between the solutions of these models thereby rationalising the indirect model.

We start by using the Bayesian game generated by a sealed bid auction to derive the parameters G and B. Suppose I is the finite set of Bidder 1's rivals, T_i is Bidder i's type space, μ_i is the prior distribution over T_i, and Bidder i's bidding strategy is the probability transition kernel $f_i : T_i \times \mathcal{B}([0, \beta_1]) \to [0, 1]$.[7] Hence, Bidder 1's belief about Bidder i's bid is the distribution $\int_{T_i} \mu_i(dt_i)\, f_i(t_i, .) \in \Delta([0, \beta_1])$ with distribution function G_i. As the rivals' bidding strategies are independent, the rival distribution function G is given by

$$G(x) = \prod_{i \in I} G_i(x) \tag{10}$$

for $x \in \Re$, which corresponds to the distribution of the highest bid amongst Bidder 1's rivals.

If Bidder 1's type is $t_1 \in T_1$ and the profile of rival bidders' types is $t \in T := \prod_{i \in I} T_i$, then let Bidder 1's gross utility be $u(t_1, t)$ if she wins the prize, and 0 otherwise. Therefore, at the interim stage, Bidder 1's (expected) valuation of the prize, contingent on her type t_1, is

$$B(t_1) = \int_T \mu(dt)\, u(t_1, t) \tag{11}$$

where $\mu = \prod_{i \in I} \mu_i$ is the prior joint distribution of the rivals' types; we shall suppress t_1 henceforth and denote $B(t_1)$ by B.

[7]This means $f_i(., E) : T_i \to [0, 1]$ is measurable for every $E \in \mathcal{B}([0, \beta_1])$ and $f_i(t_i, .) \in \Delta([0, \beta_1])$ for every $t_i \in T_i$. The interpretation of f_i is that, knowing her type $t_i \in T_i$, Bidder i will randomise her bids as *per* $f_i(t_i, .)$.

Given G and B, we specify various versions of each model, parametrised by the auction form and the setting.

1. **Indirect model**. In the σ-additive setting, with e_1 and e_2 given by (6), Bidder 1's expected payoffs in the first-price and second-price auctions are

$$V_1(B,c) = cB - e_1(c) \quad \text{and} \quad V_2(B,c) = cB - e_2(c) \tag{12}$$

respectively. In the finitely additive setting, with η_1 and η_2 given by (7), Bidder 1's expected payoffs in the first-price and second-price auctions are

$$V_1(B,c) = cB - \eta_1(c) \quad \text{and} \quad V_2(B,c) = cB - \eta_2(c) \tag{13}$$

respectively.

Bidder 1's problem in the indirect model of the first-price (resp., second-price) auction is to choose $c \in C$ to maximise $V_1(B,.)$ (resp., $V_2(B,.)$), which is given by (12) or (13) as *per* the setting.

Bidder 1's objective function, given variously by (8), (9), (12), and (13), is easily justified. For instance, if $c \in C$ and $\lambda \in \Delta(C,c)$, then $m(\lambda) = c$ and $\int_C \lambda(dy)\,[yB - g(y)] = cB - \int_C \lambda(dy)\,g(y) \leq cB - e(c)$. Thus, Bidder 1 can do no better than to choose a mean winning probability c that maximises $cB - e(c)$ and is implemented by some $\lambda \in \Delta(C,c)$.

2. **Intermediate model**. Given a distribution λ over C, Bidder 1's mean winning probability for both auction forms is

$$m(\lambda) = \int_C \lambda(dc)\, c \tag{14}$$

the expected payment in the first-price auction is

$$\hat{C}_1(\lambda) = \int_C \lambda(dc)\, b(c) G \circ b(c) \tag{15}$$

the expected payment in the second-price auction is

$$\hat{C}_2(\lambda) = \int_C \lambda(dc) \int_{[0,b(c)]} x\, dG(x) \tag{16}$$

and the expected payoffs in the first-price and second-price auctions are

$$\hat{U}_1(\lambda) = m(\lambda)B - \hat{C}_1(\lambda) \quad \text{and} \quad \hat{U}_2(\lambda) = m(\lambda)B - \hat{C}_2(\lambda) \tag{17}$$

respectively. The integrals in (14)-(16) are understood to be Lebesgue-Stieltjes integrals when $\lambda \in \Delta(C)$ in the σ-additive setting and Dunford-Schwartz integrals when $\lambda \in P(C)$ in the finitely additive setting.

Bidder 1's problem in the intermediate model of the first-price (resp., second-price) auction is to choose λ to maximise $\hat{U}_1$ (resp., $\hat{U}_2$), where λ is chosen from $\Delta(C)$ or $P(C)$ as *per* the setting.

3. **Direct model**. Let H_ν be the distribution function corresponding to the bid distribution $\nu \in \Delta([0, \beta_1])$. Given ν, the mean winning probability for both auction forms is

$$P(\nu) = \int_{[0,\beta_1]} G(x)\, dH_\nu(x) \tag{18}$$

the expected payment in the first-price auction is

$$C_1(\nu) = \int_{[0,\beta_1]} \left[\int_{[x,\beta_1]} z\, dH_\nu(z) \right] dG(x) \tag{19}$$

the expected payment in the second-price auction is

$$C_2(\nu) = \int_{[0,\beta_1]} x[1 - H_\nu(x)]\, dG(x) \tag{20}$$

and the expected utilities are

$$U_1(\nu) = P(\nu)B - C_1(\nu) \quad \text{and} \quad U_2(\nu) = P(\nu)B - C_2(\nu) \tag{21}$$

for the first-price and second-price auctions respectively.

Bidder 1's problem in the direct model of the first-price (resp., second-price) auction is to maximise U_1 (resp., U_2) over $\Delta([0, \beta_1])$. Since P, C_1, and C_2 are affine functions, so are U_1 and U_2. Therefore, the set of maximisers of U_1 (resp., U_2) is convex.

The direct model is considered in the σ-additive setting only since we do not have a dual characterisation of it in the finitely additive setting.

For a given setting and a given auction form, a duality between two of the above three models is a function from the choice space of one model in the pair to the other model's choice space such that, for every admissible rival distribution G and Bidder 1's valuation $B \in \Re_+$,

1. *the image of a solution of Bidder 1's problem in the first model solves Bidder 1's problem in the second model, and*

2. *a solution of Bidder 1's problem in the second model has a pre-image that solves Bidder 1's problem in the first model.*

The other duality results are demonstrated for the σ-additive model and require some enabling assumptions. The second duality result, Theorem 5.6, provides an equivalence mapping between the choice variables in Problems 2 and 3, namely distributions over winning probabilities and bids respectively. The third duality result, Theorem 5.8, shows that equivalent choice variables in the two auction forms lead to the same mean winning probabilities, expected payments, and expected utilities for Bidder 1. The key corollary of Theorem 5.8 is that the equivalence mapping preserves Bidder 1's preference ordering. The final duality result, Corollary 5.10, shows that a distribution over winning probabilities solves Problem 2 if and only if the equivalent bid distribution solves Problem 3.

5.1 Indirect and intermediate model dualities

We first show a duality between the indirect and intermediate models of the first-price auction in the finitely additive setting.

Theorem 5.1 *Consider $B \geq 0$, and G, η_1, V_1, m, $\hat{C}_1$, and $\hat{U}_1$ as per Equations (1), (7), (13), (14), (15), and (17), respectively.*

(A) If $\lambda \in \arg\max_{P(C)} \hat{U}_1(.)$, then $m(\lambda) \in \arg\max_C V_1(B,.)$ and $\hat{C}_1(\lambda) = \eta_1 \circ m(\lambda)$.

(B) If $c^* \in \arg\max_C V_1(B,.)$, then there exists $\lambda^* \in P(C, c^*)$ such that $\lambda^* \in \arg\max_{P(C)} \hat{U}_1(.)$.

Proof. Consider B, G, η_1, V_1, m, $\hat{C}_1$, and $\hat{U}_1$ *ex hypothesi*. Clearly, m is surjective.

(A) Consider $\lambda \in \arg\max_{P(C)} \hat{U}_1(.)$. Suppose $\hat{C}_1(\lambda) \neq \eta_1 \circ m(\lambda)$. Since $\lambda \in P(C, m(\lambda))$, we have $\hat{C}_1(\lambda) \geq \eta_1 \circ m(\lambda)$. Hence, $\hat{C}_1(\lambda) > \eta_1 \circ m(\lambda)$. So, $\hat{C}_1(\lambda) > \hat{C}_1(\lambda')$ for some $\lambda' \in P(C, m(\lambda))$. Then, $m(\lambda') = m(\lambda)$ and $\hat{U}_1(\lambda') = m(\lambda')B - \hat{C}_1(\lambda') > m(\lambda)B - \hat{C}_1(\lambda) = \hat{U}_1(\lambda)$, which is a contradiction. So, $\hat{C}_1(\lambda) = \eta_1 \circ m(\lambda)$.

Consider $c \in C$. Lemma 3.6 implies $\eta_1(c) = \hat{C}_1(\lambda_c)$ for some $\lambda_c \in P(C, c)$. Hence, $m(\lambda_c) = c$ and $V_1(B, m(\lambda)) = m(\lambda)B - \eta_1 \circ m(\lambda) = m(\lambda)B - \hat{C}_1(\lambda) = \hat{U}_1(\lambda) \geq \hat{U}_1(\lambda_c) = m(\lambda_c)B - \hat{C}_1(\lambda_c) = cB - \eta_1(c) = V_1(B, c)$. So, $m(\lambda) \in \arg\max_C V_1(B,.)$.

(B) Consider $c^* \in \arg\max_C V_1(B,.)$. Lemma 3.6 implies $\eta_1(c^*) = \hat{C}_1(\lambda^*)$ for some $\lambda^* \in P(C, c^*)$. Consider $\lambda \in P(C)$. Then, $\eta_1 \circ m(\lambda) \leq \hat{C}_1(\lambda)$ and $\hat{U}_1(\lambda^*) = m(\lambda^*)B - \hat{C}_1(\lambda^*) = c^*B - \eta_1(c^*) = V_1(B, c^*) \geq V_1(B, m(\lambda)) = m(\lambda)B - \eta_1 \circ m(\lambda) \geq m(\lambda)B - \hat{C}_1(\lambda) = \hat{U}_1(\lambda)$. So, $\lambda^* \in \arg\max_{P(C)} \hat{U}_1(.)$. ■

By analogous arguments, we have this duality in the σ-additive setting.

Theorem 5.2 *Consider $B \geq 0$, and G, e_1, V_1, m, $\hat{C}_1$, and $\hat{U}_1$ as per Equations (1), (6), (12), (14), (15), and (17), respectively. Suppose G is continuous and strictly increasing.*

(A) If $\lambda \in \arg\max_{\Delta(C)} \hat{U}_1(.)$, then $m(\lambda) \in \arg\max_C V_1(B,.)$ and $\hat{C}_1(\lambda) = e_1 \circ m(\lambda)$.

(B) If $c^* \in \arg\max_C V_1(B,.)$, then there exists $\lambda^* \in \Delta(C, c^*)$ such that $\lambda^* \in \arg\max_{\Delta(C)} \hat{U}_1(.)$.

Remark 5.3 *By analogous arguments, the above two results also hold for the second-price auction, with e_1, η_1, $\hat{C}_1$, V_1, and $\hat{U}_1$ replaced by e_2, η_2, $\hat{C}_2$, V_2, and $\hat{U}_2$ respectively. The analogue of Theorem 5.2 for the second-price auction requires G to be only continuous.*

The conditions on G in Theorem 5.2 (resp., Remark 5.3) ensure that g_1 (resp., g_2) is continuous as *per* Lemma 3.3. This continuity is required in the proofs for the application of Lemma 3.5 in place of Lemma 3.6.

The following argument demonstrates the canonicity of the indirect formulation by showing that the choice of an equivalence class of distributions in the intermediate formulation is generally reducible to an elementary choice of a probability in the unit interval.

Remark 5.4 *Let $C' = \arg\max_C V_1(B,.)$ in Theorem 5.1's setting. By Theorem 5.1(B), for every $c \in C'$, there exists $\lambda_c \in P(C,c)$ such that $\lambda_c \in \arg\max_{P(C)} \hat{U}_1(.)$. Hence, $\hat{U}_1$ is constant over $\{\lambda_c \mid c \in C'\}$. More generally, $\hat{U}_1$ is constant over $A := \cup_{c \in C'}\{\lambda \in P(C) \mid \hat{U}_1(\lambda_c) = \hat{U}_1(\lambda)\}$. Consequently, A is an equivalence class and $A \subset \arg\max_{P(C)} \hat{U}_1(.)$.*

Conversely, consider $\lambda \in \arg\max_{P(C)} \hat{U}_1(.)$. By Theorem 5.1(A), $m(\lambda) \in \arg\max_C V_1(B,.) = C'$. Hence, $\lambda_{m(\lambda)} \in \arg\max_{P(C)} \hat{U}_1(.)$. As $\hat{U}_1(\lambda_{m(\lambda)}) = \hat{U}_1(\lambda)$ and $m(\lambda) \in C'$, we have $\lambda \in A$. Hence, $A = \arg\max_{P(C)} \hat{U}_1(.)$.

Thus, C' is interpretable as the canonical representation of the equivalence class $\cup_{c \in C'}\{\lambda \in P(C) \mid \hat{U}_1(\lambda_c) = \hat{U}_1(\lambda)\}$.

5.2 Intermediate and direct model dualities

We now turn to the dualities between models involving the intermediate and the direct formulations in the σ-additive setting. Unlike the above dualities, this will require some extra regularity constraints on G and the choices available to Bidder 1 in these models.

The first step shows that, if G is continuous, then the search for optimal bid distributions may be restricted to $\Delta(b(C))$ without loss of generality.

Lemma 5.5 *If G is continuous and $\nu \in \Delta([0, \beta_1]) \setminus \Delta(b(C))$, then there exists $\nu' \in \Delta(b(C))$ such that $U_1(\nu') > U_1(\nu)$ and $U_2(\nu') > U_2(\nu)$.*

The second result provides the candidate for the duality mapping. Let $\Gamma(\lambda) = \lambda \circ b^{-1}$ for $\lambda \in \Delta(C)$; clearly, $\Gamma(\lambda) \in \Delta(b(C))$.

Lemma 5.6 *Suppose G is continuous. Then,*

(A) $\Gamma : \Delta(C) \to \Delta(b(C))$ is a bijection with function inverse Γ^{-1}, and

(B) if G is also strictly increasing, then Γ is a homeomorphism from $\Delta(C)$ to $\Delta([0, \beta_1])$, with $\Delta(C)$ and $\Delta([0, \beta_1])$ given their weak topologies.*

The following lemma provides the key technical tool for relating Bidder 1's expected payoffs from the pairs of distributions $(\lambda, \Gamma(\lambda))$. Henceforth, F_λ and H_ν denote the distribution functions corresponding to $\lambda \in \Delta(C)$ and $\nu \in \Delta([0, \beta_1])$ respectively. Given $H_{\Gamma(\lambda)}$, define $h_{\Gamma(\lambda)} : [0, \beta_1] \to \Re$ by $h_{\Gamma(\lambda)}(x) = x H_{\Gamma(\lambda)}(x)$.

Lemma 5.7 *Consider G and $\lambda \in \Delta(C)$. If*

(a) G and F_λ are continuous, and

(b) G has a bounded derivative on $[0, \beta_1] \setminus M$ and F_λ has a bounded derivative on $C \setminus L$, where M and L are countable,

then G, F_λ, $H_{\Gamma(\lambda)}$, and $h_{\Gamma(\lambda)}$ are absolutely continuous.

As G and F_λ are distribution functions, they are right-continuous and increasing, and therefore, differentiable Leb-a.e. Hypothesis (a) strengthens right-continuity to continuity and hypothesis (b) strengthens Leb-a.e. differentiability to bounded differentiability on co-countable sets.[8] Applying this result, we show that Γ preserves mean winning probabilities, expected payments, and expected payoffs.

Theorem 5.8 *If G and $\lambda \in \Delta(C)$ meet the hypotheses of Lemma 5.7, then*

(A) $\hat{C}_1(\lambda) = C_1(\Gamma(\lambda))$ and $\hat{C}_2(\lambda) = C_2(\Gamma(\lambda))$,

(B) $m(\lambda) = P(\Gamma(\lambda))$, and

(C) $\hat{U}_1(\lambda) = U_1(\Gamma(\lambda))$ and $\hat{U}_2(\lambda) = U_2(\Gamma(\lambda))$.

It follows immediately that Γ is order-preserving on $\Delta(C)$.

[8]Distribution functions ruled out by hypothesis (b) include the Cantor function. While it meets hypothesis (a), it does not satisfy hypothesis (b) as it is non-differentiable on the entire Cantor set, which is uncountable.

Corollary 5.9 *If G and $\lambda_1, \lambda_2 \in \Delta(C)$ meet the hypotheses of Lemma 5.7, then $\hat{U}_1(\lambda_1) \geq \hat{U}_1(\lambda_2)$ if and only if $U_1(\Gamma(\lambda_1)) \geq U_1(\Gamma(\lambda_2))$, and $\hat{U}_2(\lambda_1) \geq \hat{U}_2(\lambda_2)$ if and only if $U_2(\Gamma(\lambda_1)) \geq U_2(\Gamma(\lambda_2))$.*

We finally have the duality between the solutions of Bidder 1's problem in the intermediate and direct models of the first-price auction in the σ-additive setting. Let $\Delta(C)_0$ be the set of $\lambda \in \Delta(C)$ that satisfy the hypotheses of Lemma 5.7. Let $\Delta([0, \beta_1])_0 = \Gamma(\Delta(C)_0)$. Using Lemma 5.6, if G is continuous, then Γ is a bijection from $\Delta(C)_0$ to $\Delta([0, \beta_1])_0$.

Corollary 5.10 *Consider the first-price auction. If G and λ meet the hypotheses of Lemma 5.7, then $\hat{U}_1(\lambda) \geq \hat{U}_1(\lambda')$ for every $\lambda' \in \Delta(C)_0$ if and only if $U_1(\Gamma(\lambda)) \geq U_1(\nu)$ for every $\nu \in \Delta([0, \beta_1])_0$.*

The analogous result holds for the second-price auction with $\hat{U}_2$, U_2, and $\hat{C}_2$ replacing $\hat{U}_1$, U_1, and $\hat{C}_1$ respectively.

6 Concluding remarks

We have analysed models of bidding behaviour in first-price and second-price sealed bid auctions of a prize wherein, given the rivals' bid distributions, a bidder chooses a mean winning probability that is generated by a distribution over winning probabilities or equivalently – as shown by our duality results – by a bid distribution. The mean winning probabilities chosen in these models increase – indicating greater aggression – if and only if the bidder's valuation of the prize increases.

This formulation of bidding behaviour has obvious advantages. First, bidding behaviour is represented simply by just the first moment of a distribution over winning probabilities, i.e., a single number from the unit interval, instead of an entire distribution. Consequently, an optimal choice has an elementary characterisation. Secondly, the variational analysis and result are simplified and sharpened by the properties of the usual ordering on $\Re$. A similar analysis of chosen distributions would have to rely on partial stochastic orders.

It is natural to ask if bidding behaviour in terms of bid distributions or distributions over winning probabilities bears any relation to the model of bidding behaviour in terms of mean winning probabilities. Given some regularity conditions, we have shown that these three models of bidding behaviour are equivalent if attention is restricted to σ-additive distributions over winning probabilities and σ-additive bid distributions. Moreover, it is shown in great generality that the models of bidding behaviour in terms of distributions over winning probabilities and mean winning probabilities are equivalent for finitely additive and σ-additive distributions over winning probabilities.

An obvious complementary question for future investigation is the effect of variations in the rival distribution function G on bidding behaviour.

Appendix

Proof of Lemma 3.1 Consider b and G *ex hypothesi*.

(A) Clearly, b is bounded and nonnegative.

If $c_1, c_2 \in C$ and $c_1 < c_2$, then $G^{-1}([c_2, 1]) \subset G^{-1}([c_1, 1])$. Consequently, $b(c_1) \leq b(c_2)$, i.e., b is increasing and therefore measurable.

If $\alpha < 0$, then $b^{-1}((\alpha, \infty)) = C$, which is open in C. Consider $\alpha \geq 0$. If $b^{-1}((\alpha, \infty)) = \emptyset$, then $b^{-1}((\alpha, \infty))$ is open in C. Suppose there exists $c \in b^{-1}((\alpha, \infty))$. Then, $c \in (0, 1]$ and $\inf G^{-1}([c, 1]) = b(c) > \alpha$. Consequently, $G(\alpha) < c$. Thus, $b^{-1}((\alpha, \infty)) \subset (0, 1] \cap (G(\alpha), \infty)$. Conversely, suppose $c \in (0, 1] \cap (G(\alpha), \infty)$. As $G(b(c)) \geq c$, we have $G(\alpha) < G(b(c))$. As G is increasing, $\alpha < b(c)$. Thus, $b^{-1}((\alpha, \infty)) \supset (0, 1] \cap (G(\alpha), \infty)$. So, $b^{-1}((\alpha, \infty)) = (0, 1] \cap (G(\alpha), \infty)$, which is open in C. Therefore, b is left-continuous.

(B) Suppose G is not strictly increasing on $[0, \beta_1]$. Then, there exist $b_1, b_2 \in [0, \beta_1]$ such that $b_1 < b_2$ and $G(b_1) = G(b_2) = c$. As $b_1 < b_2 \leq \beta_1 = \inf G^{-1}(\{1\})$, we have $c = G(b_1) < 1$. So, there exists $N \in \mathcal{N}$ such that $c + 1/N < 1$. For every $n \geq N$, we have $b(c + 1/n) \geq b_2 > b_1 \geq b(c)$. Thus, $b(c+) \geq b_2 > b(c)$, i.e., b is not continuous.

Conversely, suppose b is not continuous. As b is increasing and left-continuous, there exists $c \in [0, 1)$ and $\alpha_1, \alpha_2 \in \Re$ such that $b(c) < \alpha_1 < \alpha_2 < b(c+) \leq b(c + 1/n)$ for every $n \in \mathcal{N}$ such that $c + 1/n \leq 1$. As G is increasing, we have $c \leq G(\alpha_1) \leq G(\alpha_2) < c + 1/n$. It follows that $G(\alpha_1) = c = G(\alpha_2)$. Thus, G is not strictly increasing.

Suppose b is not strictly increasing. Then, there exist $c_1, c_2 \in C$ such that $c_1 < c_2$ and $b(c_1) = b(c_2)$. Then, $G(b(c_1)) = G(b(c_2)) \geq c_2 > c_1$. If $c_1 = 0$, then $G(0) = G(b(0)) > 0 = G(0-)$, i.e., G is not continuous. If $c_1 > 0$, then $G(b(c_1) - 1/n) < c_1$ for every $n \in \mathcal{N}$, which implies $G(b(c_1)) > c_1 \geq G(b(c_1)-)$, i.e., G is not continuous.

Conversely, suppose G is not continuous. As G is increasing and right-continuous, there exists $\alpha \in (0, \beta_1]$ and $c_1, c_2 \in (0, 1)$ such that $G(\alpha) > c_2 > c_1 > G(\alpha-)$. As G is increasing, $[\alpha, \infty) \subset G^{-1}([c_1, 1])$. If $\beta < \alpha$, then $G(\beta) \leq G(\alpha-) < c_1$, which means $\beta \notin G^{-1}([c_1, 1])$. So, $[\alpha, \infty) \supset G^{-1}([c_1, 1])$. Thus, $G^{-1}([c_1, 1]) = [\alpha, \infty)$. Similarly, $G^{-1}([c_2, 1]) = [\alpha, \infty)$. By definition, $b(c_1) = \alpha = b(c_2)$. Thus, b is not strictly increasing.

(C) By definition, there exists an increasing sequence $(c_n) \subset b^{-1}(\{b(c)\})$ converging to $\bar{c}$. As b is left-continuous by (A) and $b(c_n) = b(c)$ for every c_n, we have $b(\bar{c}) = \lim_n b(c_n) = b(c)$.

(D) Consider $c \in C$. Then, $\bar{c} \in C$. As G is right-continuous, $b(\bar{c}) \in G^{-1}([G \circ b(\bar{c}), 1]) \cap [0, \beta_1]$. Therefore, $b(\bar{c}) \geq \inf[G^{-1}([G \circ b(\bar{c}), 1]) \cap [0, \beta_1]] = b \circ G \circ b(\bar{c}) \geq b(\bar{c})$ as $G \circ b(c) \geq c$ for every $c \in C$ and b is increasing by (A). So,

$$b(\bar{c}) = b \circ G \circ b(\bar{c}) \tag{22}$$

Suppose there exists $c' \in C$ such that $c' > \bar{c}$. As b is increasing by (A), $b(c') \geq b(\bar{c})$. Suppose $b(c') = b(\bar{c})$. As $b(\bar{c}) = b(c)$ by (C), we have $b(c') = b(c)$. Hence, $c' \in b^{-1}(\{b(c)\})$. This implies $c' > \bar{c} = \sup b^{-1}(\{b(c)\}) \geq c'$, which is a contradiction. Hence, $b(c') > b(\bar{c})$.

Since $\bar{c} \in C$ and $G \circ b(c) \geq c$ for every $c \in C$, we have $G \circ b(\bar{c}) \geq \bar{c}$. If $G \circ b(\bar{c}) > \bar{c}$, then $b \circ G \circ b(\bar{c}) > b(\bar{c})$, which contradicts (11). So, $\bar{c} = G \circ b(\bar{c}) \in G([0, \beta_1])$.

(E) Combining (C) and (D), $c < \bar{c}$ implies $c < \bar{c} = G \circ b(\bar{c}) = G \circ b(c)$.

(F) It follows from (D) that $G([0, \beta_1]) \supset \{c \in C \mid c = \bar{c}\}$.

Conversely, consider $c \in G([0, \beta_1])$. Then, $G^{-1}(\{c\}) \cap [0, \beta_1] \neq \emptyset$.

If $|G^{-1}(\{c\}) \cap [0, \beta_1]| = 1$, then there is a unique $\beta \in [0, \beta_1]$ such that $G(\beta) = c$. As G is increasing, $b(c) = \inf[G^{-1}([c, 1]) \cap [0, \beta_1]] = \inf[\beta, \beta_1] = \beta$. So, $c \in b^{-1}(\{\beta\}) = b^{-1}(\{b(c)\})$. Suppose there exists $c' \in C$ such that $c' > c$ and $c' \in b^{-1}(\{b(c)\})$. Then, $b(c') = \beta \notin G^{-1}([c', 1])$ since $G(\beta) = c < c'$. As G is right-continuous, $\beta < \inf G^{-1}([c', 1])$. So, $b(c') = \inf[G^{-1}([c', 1]) \cap [0, \beta_1]] > \beta$, a contradiction. It follows that $c = \sup b^{-1}(\{b(c)\}) = \bar{c}$.

Suppose $|G^{-1}(\{c\}) \cap [0, \beta_1]| > 1$. As $G(0) = 0$ and $G(b) > 0$ for $b > 0$, we have $|G^{-1}(\{0\}) \cap [0, \beta_1]| = 1$. So, $c > 0$ and $b(c) = \inf[G^{-1}([c, 1]) \cap [0, \beta_1]] = \inf G^{-1}([c, 1])$. The right-continuity of G implies the existence of $\underline{\beta}, \bar{\beta} \in [0, \beta_1]$ such that $\underline{\beta} < \bar{\beta}$ and $G^{-1}(\{c\}) = [\underline{\beta}, \bar{\beta})$. As G is increasing, $b(c) = \inf G^{-1}([c, 1]) = \inf[\underline{\beta}, \beta_1] = \underline{\beta}$. So, $c \in b^{-1}(\{\underline{\beta}\}) = b^{-1}(\{b(c)\})$. Suppose there exists $c' \in C$ such that $c' > c$ and $c' \in b^{-1}(\{b(c)\})$. Then, $c' > 0$ and $b(c') = b(c) = \underline{\beta} \notin G^{-1}([c', 1])$ since $G(\underline{\beta}) = c$. Therefore, $b(c) = b(c') = \inf[G^{-1}([c', 1]) \cap [0, \beta_1]] = \inf G^{-1}([c', 1]) > \underline{\beta}$, which is a contradiction. It follows that $c = \sup b^{-1}(\{b(c)\}) = \bar{c}$. Hence, $G([0, \beta_1]) \subset \{c \in C \mid c = \bar{c}\}$.

In addition, consider $\beta \in [0, \beta_1]$. Then, $G(\beta) \in G([0, \beta_1])$. Consequently, $G(\beta) = \overline{G(\beta)}$. Therefore, $G \circ b \circ G(\beta) = G \circ b(\overline{G(\beta)}) = \overline{G(\beta)} = G(\beta)$.

(G) Consider $c \in b^{-1}([0,x])$. Then, $b(c) \in [0,x]$. By (B), b is strictly increasing. So, $c = \bar{c}$. Using (D), $c = G \circ b(\bar{c}) = G \circ b(c) \in G([0,x])$. Conversely, consider $c \in G([0,x])$. Then, $c = G(y)$ for some $y \in [0,x]$. Using (2), $b(c) \in [0,y] \subset [0,x]$. So, $c \in b^{-1}([0,x])$. ■

Proof of Lemma 3.2 Consider either auction form. As b is nonnegative and G is a distribution function, g is nonnegative. As $b(0) = 0$, we have $g(0) = 0$. As b is bounded and G is a distribution function, g is bounded. As b and G are increasing, so is g. As g is increasing and bounded, it is integrable. ■

Proof of Lemma 3.3 Consider G, b, g_1, and g_2 *ex hypothesi*.

(A) As G is strictly increasing, it is injective. As b is strictly increasing, $c = \bar{c}$ for every $c \in C$. By Lemma 3.1, G is surjective and G (resp., b) is the function inverse of b (resp., G). G is continuous by assumption and b is continuous by Lemma 3.1.

(B) If G is continuous and strictly increasing, then so is b by Lemma 3.1. Hence, g_1 is continuous. The converse is proved as follows.

(a) Suppose G is discontinuous at $\beta \in (0, \beta_1)$ with $G(\beta) = c$. Then, $G(\beta-) = \underline{c} < c = G(\beta)$. As $b = \beta$ on $(\underline{c}, c]$, we have $G \circ b(c') = G(\beta)$ for $c' \in (\underline{c}, c]$. Hence, $b(\underline{c}+) = \beta$, $G \circ b(\underline{c}+) = G(\beta)$, and $g_1(\underline{c}+) = b(\underline{c}+)G \circ b(\underline{c}+) = \beta G(\beta)$. As b is left-continuous by Lemma 3.1, $b(\underline{c}-) = b(\underline{c}) \leq \beta$.
Let $b(\underline{c}) < \beta$. Then, $G \circ b(\underline{c}-) \leq G(\beta-)$ and $g_1(\underline{c}+) = \beta G(\beta) > b(\underline{c}-)G(\beta-) \geq b(\underline{c}-)G \circ b(\underline{c}-) = g_1(\underline{c}-)$.
Let $b(\underline{c}) = \beta$. Consider $n \in \mathcal{N}$. By the definition of $\underline{c}$, we have $b(\underline{c} - 1/n) < b(\underline{c}) = \beta$. So, there exists $\delta > 0$ such that $b(\underline{c} - 1/n) < \beta - \delta$ and $G \circ b(\underline{c} - 1/n) \leq G(\beta - \delta) \leq G(\beta-)$. Hence, $G \circ b(\underline{c}-) \leq G(\beta-)$. It follows that $g_1(\underline{c}+) = \beta G(\beta) > b(\underline{c}-)G(\beta-) \geq b(\underline{c}-)G \circ b(\underline{c}-) = g_1(\underline{c}-)$.
So, g_1 is discontinuous at $\underline{c}$ in both cases.

(b) Suppose G is not strictly increasing on $[0, \beta_1]$. Then, there exists $c \in (0,1)$ and $\beta \in (0, \beta_1)$ such that $b(c) < \beta$ and $G(\beta) = c$. As b is left-continuous by Lemma 3.1, $b(c-) = b(c)$. If G is continuous at $b(c)$, then $g_1(c-) = b(c-)G \circ b(c-) = b(c)G \circ b(c)$. If G is discontinuous at $b(c)$, then $\underline{c} < c$ and $b = b(c)$ on $(\underline{c}, c]$. So, $g_1(c-) = b(c-)G \circ b(c-) = b(c)G \circ b(c)$. Also, $b(c+) \geq \beta > b(c)$ and $g_1(c+) = b(c+)G \circ b(c+) > b(c)G \circ b(c) = g_1(c-)$. Hence, g_1 is discontinuous at c.

(C) Suppose G is continuous. Consider $c \in C$. As b is left-continuous by Lemma 3.1, $b(c+) \geq b(c) = b(c-)$. Applying the monotone convergence theorem (Bruckner et al. [2], Theorem 5.8) to (5), we have

$g_2(c+) = b(c+)G \circ b(c+) - \int_{[0,b(c+)]} dy\, G(y)$ and $g_2(c-) = b(c)G \circ b(c) - \int_{[0,b(c)]} dy\, G(y)$. As G is continuous, b is strictly increasing, and so $b(c+) = \max G^{-1}(\{c\})$. Therefore, $G \circ b(c+) = c$. By Lemma 3.1, as b is strictly increasing, $c = \bar{c}$ and $G \circ b(c) = G \circ b(\bar{c}) = \bar{c} = c$. So, $g_2(c+) - g_2(c-) = [b(c+) - b(c)]c - \int_{(b(c),b(c+)]} dy\, G(y) = [b(c+) - b(c)]c - [b(c+) - b(c)]c = 0$, i.e., g_2 is continuous at c.

Conversely, suppose G is discontinuous at some $\beta \in [0, \beta_1]$. By (1), $\beta \in (0, \beta_1)$. Let $c := G(\beta) > G(\beta-)$. By (1) and (3), $\bar{c} > \underline{c} > 0$ and $\beta = b(c') \geq b(\underline{c})$ for every $c' \in (\underline{c}, \bar{c}]$. Consider $n \in \mathcal{N}$ such that $\underline{c} - 1/n, \underline{c} + 1/n \in [0, \bar{c}]$. By (3), $b(\underline{c} - 1/n) < b(\underline{c})$ and $b(\underline{c} + 1/n) = \beta$. As b is left-continuous by Lemma 3.1, $\lim_n b(\underline{c} - 1/n) = b(\underline{c})$. So, $1_{[0,b(\underline{c}+1/n)]} - 1_{[0,b(\underline{c}-1/n)]} = 1_{(b(\underline{c}-1/n),\beta]}$, and by (4), $g_2(\underline{c} + 1/n) - g_2(\underline{c} - 1/n) = \int_{\Re} y 1_{(b(\underline{c}-1/n),\beta]}(y)\, dG(y)$. As $\lim_n 1_{(b(\underline{c}-1/n),\beta]} = 1_{[b(\underline{c}),\beta]}$, the monotone convergence theorem implies $g_2(\underline{c}+) - g_2(\underline{c}-) = \lim_n \int_{\Re} y 1_{(b(\underline{c}-1/n),\beta]}(y)\, dG(y) = \int_{\Re} y \lim_n 1_{(b(\underline{c}-1/n),\beta]}(y)\, dG(y)$, which equals $\int_{\Re} y 1_{[b(\underline{c}),\beta]}(y)\, dG(y) \geq \int_{\Re} y 1_{\{\beta\}}(y)\, dG(y) = \beta[G(\beta) - G(\beta-)] > 0$, i.e., g_2 is discontinuous at $\underline{c}$. ■

Proof of Lemma 3.4 Consider g *ex hypothesi*.

(A) Consider $c \in C$ and $\lambda \in \Delta(C, c)$. So, $\delta_c \in \Delta(C, c)$. As g is convex, $g(c) = g(\int_C \lambda(dx)\, x) \leq \int_C \lambda(dx)\, g(x)$. Hence, $g(c) = L(g, \delta_c) = \min\{L(g, \lambda) \mid \lambda \in \Delta(C, c)\} = e(c)$.

(B) follows from (A) and Lemma 3.2.

(C) As g is convex, g is continuous on $(0, 1)$ (Aliprantis and Border [1], Theorem 7.22). As g is increasing, $g(0) \leq \inf g((0, 1])$. As g is convex, $g(0) \geq \inf g((0, 1])$. Thus, $g(0) = \inf g((0, 1])$ and g is continuous at 0. So, g is continuous on $[0, 1)$.

(D) As g is convex, it is subdifferentiable on $(0, 1)$ (Aliprantis and Border [1], Theorem 7.23) and differentiable on $(0, 1) \setminus E$ where E is countable (Aliprantis and Border [1], Theorem 7.22). By the Busemann-Feller theorem, it is twice differentiable Leb-a.e. on $(0, 1)$.

(E) follows from (B) and (C). ■

Proof of Lemma 3.5 Consider g *ex hypothesi*.

(A) Give $\Delta(C)$ its weak* topology. It follows that $\Delta(C)$ is a compact metric space (Parthasarathy [6], Theorem II.6.4) and $L(g, .)$ is continuous.

Let $\lambda \in \Delta(C)$ be an accumulation point of $\Delta(C, c)$. Then, there is a sequence $(\lambda_n) \subset \Delta(C, c)$ converging to λ. As the identity mapping on C

is continuous, $m(\lambda) = \int_C \lambda(dx)\, x = \lim_n \int_C \lambda_n(dx)\, x = \lim_n m(\lambda_n) = c$. So, $\lambda \in \Delta(C, c)$. Hence, $\Delta(C, c)$ is closed.

Therefore, $\Delta(C, c)$ is compact and there exists $\lambda_c \in \Delta(C, c)$ such that $L(g, \lambda_c) = \min\{L(g, \lambda) \mid \lambda \in \Delta(C, c)\}$.

(B) As $L(g, .)$ is bounded and nonnegative, so is e. As $\delta_0 \in \Delta(C, 0)$, we have $0 \leq e(0) \leq L(g, \delta_0) = g(0) = 0$. Hence, $e(0) = 0$.

(C) Consider $c_1, c_2 \in C$, $t \in (0, 1)$, and $c = tc_1 + (1 - t)c_2$. By (1), there exists $\lambda_1 \in \Delta(C, c_1)$ and $\lambda_2 \in \Delta(C, c_2)$ such that $e(c_1) = L(g, \lambda_1)$ and $e(c_2) = L(g, \lambda_2)$. Set $\lambda = t\lambda_1 + (1 - t)\lambda_2$. Then, $\lambda \in \Delta(C, c)$ as $\lambda \in \Delta(C)$ and $m(\lambda) = tm(\lambda_1) + (1-t)m(\lambda_2) = tc_1 + (1-t)c_2 = c$. It follows that $e(c) \leq L(g, \lambda) = tL(g, \lambda_1) + (1-t)L(g, \lambda_2) = te(c_1) + (1-t)e(c_2)$.

(D) For every $c \in C$, since $\Delta(C, c) \subset \{\lambda \in \Delta(C) \mid m(\lambda) \geq c\}$, we have $e(c) = \inf\{L(g, \lambda) \mid \lambda \in \Delta(C, c)\} \geq \inf\{L(g, \lambda) \mid \lambda \in \Delta(C) \ \wedge \ m(\lambda) \geq c\}$.

Consider $\lambda \in \Delta(C)$ with $m(\lambda) > c \geq 0$. As $c/m(\lambda) < 1$, $h(x) = xc/m(\lambda)$ yields the function $h : C \to C$. It follows that $\lambda \circ h^{-1} \in \Delta(C)$ and $\int_C \lambda \circ h^{-1}(dy)\, y = \int_C \lambda(dx)\, h(x) = c$. Thus, $\lambda \circ h^{-1} \in \Delta(C, c)$ and $L(g, \lambda \circ h^{-1}) = \int_C \lambda \circ h^{-1}(dy)\, g(y) = \int_C \lambda(dx)\, g \circ h(x) \leq \int_C \lambda(dx)\, g(x) = L(g, \lambda)$ as g is increasing and $h(x) \leq x$ for every $x \in C$.

So, for every $\lambda_0 \in \{\lambda \in \Delta(C) \mid m(\lambda) \geq c\}$, there exists $\lambda_1 \in \Delta(C, c)$ such that $L(g, \lambda_1) \leq L(g, \lambda_0)$. It follows that $e(c) = \inf\{L(g, \lambda) \mid \lambda \in \Delta(C, c)\} \leq \inf\{L(g, \lambda) \mid \lambda \in \Delta(C) \ \wedge \ m(\lambda) \geq c\}$. Consequently, $e(c) = \inf\{L(g, \lambda) \mid \lambda \in \Delta(C) \ \wedge \ m(\lambda) \geq c\}$.

If $c_1, c_2 \in C$ and $c_1 < c_2$, then $\{\lambda \in \Delta(C) \mid m(\lambda) \geq c_2\} \subset \{\lambda \in \Delta(C) \mid m(\lambda) \geq c_1\}$. Consequently, $e(c_2) = \inf\{L(g, \lambda) \mid \lambda \in \Delta(C) \ \wedge \ m(\lambda) \geq c_2\} \geq \inf\{L(g, \lambda) \mid \lambda \in \Delta(C) \ \wedge \ m(\lambda) \geq c_1\} = e(c_1)$.

(E) We first show that $\Delta(C, .) : C \to 2^{\Delta(C)}$ is upper hemicontinuous. As $\Delta(C)$ is compact, it suffices to show that $\mathrm{Gr}\, \Delta(C, .)$ is closed in $C \times \Delta(C)$. Consider a sequence $(c_n, \lambda_n) \subset \mathrm{Gr}\, \Delta(C, .)$ converging to (c, λ). It follows that $\lambda_n \in \Delta(C, c_n)$, i.e., $\int_C \lambda_n(dx)\, x = c_n$, for every n. It follows that $\int_C \lambda(dx)\, x = \lim_n \int_C \lambda_n(dx)\, x = \lim_n c_n = c$. This means $(c, \lambda) \in \mathrm{Gr}\, \Delta(C, .)$. Hence, $\mathrm{Gr}\, \Delta(C, .)$ is closed in $C \times \Delta(C)$.

Next we show that $\Delta(C, .)$ is lower hemicontinuous at $c \in C$. Consider $\lambda \in \Delta(C, c)$ and a sequence $(c_n) \subset C \setminus \{c\}$ converging to c. It suffices to construct a sequence $(\lambda_n) \subset \Delta(C)$ converging to λ such that $\lambda_n \in \Delta(C, c_n)$ for every n. For $n \in \mathcal{N}$, if $c_n < c$, then let $t_n = (c - c_n)/c$ and $\lambda_n = t_n\delta_0 + (1 - t_n)\lambda$; if $c_n > c$, then $t_n = (c_n - c)/(1 - c)$ and $\lambda_n = t_n\delta_1 + (1 - t_n)\lambda$. Then, $\int_C \lambda_n(dx)\, x = c_n$ for every n, i.e.,

$\lambda_n \in \Delta(C, c_n)$ for every n. In order to check that $\lim_n \lambda_n = \lambda$, consider $f \in \mathcal{C}(C)$. Then,

$$\int_C \lambda_n(dx)\, f(x) = \begin{cases} t_n f(0) + (1 - t_n) \int_C \lambda(dx)\, f(x), & \text{if } c_n < c \\ t_n f(1) + (1 - t_n) \int_C \lambda(dx)\, f(x), & \text{if } c_n > c \end{cases}$$

As $\lim_n t_n = 0$, $\lim_n \int_C \lambda_n(dx)\, f(x) = \int_C \lambda(dx)\, f(x)$. Thus, (λ_n) converges to λ.

It follows from the above arguments that $\Delta(C, .)$ is continuous. As $\Delta(C)$ is metrisable, it is a Hausdorff space. Moreover, as $\mathrm{Gr}\, \Delta(C, .)$ is closed in $C \times \Delta(C)$, $\Delta(C, .)$ has closed images. As $\Delta(C)$ is compact, the images are compact. As $L(g, .)$ is continuous, Berge's theorem of the maximum implies that e is continuous.

(F) This follows from (C) by copying the argument for Lemma 3.4(D).

(G) This follows from (C), (D), and (E). ■

Proof of Lemma 3.6 Consider g as *per* definition.

(A) Equip $\mathrm{ba}(C)$ with its weak* topology. As $g \in B(C)$, $L(g, .)$ is continuous. By Alaoglu's theorem (Dunford and Schwartz [3], Theorem V.4.2), the closed unit sphere of $\mathrm{ba}(C)$ is compact. As $P(C, c)$ is a subset of this sphere, if it is closed, then it is compact and the desired result follows immediately.

To verify that $P(C, c)$ is closed, consider $\lambda \in \mathrm{ba}(C)$ that is an accumulation point of $P(C, c)$. Then, there exists a net $(\lambda_n) \subset P(C, c)$ converging to λ. By definition, $\lim_n \lambda_n(E) = \lim_n \int_C \lambda_n(dx)\, 1_E(x) = \int_C \lambda(dx)\, 1_E(x) = \lambda(E)$ for every $E \in \mathcal{B}(C)$. Consequently, $\lambda(\emptyset) = 0$, $\lambda(C) = 1$ and $\lambda \geq 0$. Consider pairwise disjoint sets $E_1, \ldots, E_k \in \mathcal{B}(C)$. Then, $E = \cup_{i=1}^k E_i \in \mathcal{B}(C)$. As each λ_n is finitely additive, we have $\lambda(E) = \lim_n \lambda_n(E) = \lim_n \sum_{i=1}^k \lambda_n(E_i) = \sum_{i=1}^k \lim_n \lambda_n(E_i) = \sum_{i=1}^k \lambda(E_i)$. So, λ is finitely additive. As the identity map on C is bounded and measurable, it belongs to $B(C)$. Therefore, $c = \lim_n \int_C \lambda_n(dx)\, x = \int_C \lambda(dx)\, x$. Thus, $\lambda \in P(C, c)$. It follows that $P(C, c)$ is closed.

(B) Note that $\delta_0 \in P(C, 0)$ and copy the argument for Lemma 3.5(B).

(C) Copy the argument for Lemma 3.5(C).

(D) For every $c \in C$, as $P(C, c) \subset \{\lambda \in P(C) \mid m(\lambda) \geq c\}$, we have $e(c) = \inf\{L(g, \lambda) \mid \lambda \in P(C, c)\} \geq \inf\{L(g, \lambda) \mid \lambda \in P(C) \ \wedge \ m(\lambda) \geq c\}$.

Consider $\lambda \in P(C)$ with $m(\lambda) > c \geq 0$. As $c/m(\lambda) < 1$, $h(x) = xc/m(\lambda)$ yields the function $h : C \to C$. It follows that $\lambda \circ h^{-1} \in$

$P(C)$ and $\int_C \lambda \circ h^{-1}(dy)\, y = \int_C \lambda(dx)\, h(x) = c$. Thus, $\lambda \circ h^{-1} \in P(C, c)$ and $L(g, \lambda \circ h^{-1}) = \int_C \lambda \circ h^{-1}(dy)\, g(y) = \int_C \lambda(dx)\, g \circ h(x) \le \int_C \lambda(dx)\, g(x) = L(g, \lambda)$ as g is increasing and $h(x) \le x$ for every $x \in C$.

So, for every $\lambda_0 \in \{\lambda \in P(C) \mid m(\lambda) \ge c\}$, there exists $\lambda_1 \in P(C, c)$ such that $L(g, \lambda_1) \le L(g, \lambda_0)$. It follows that $e(c) = \inf\{L(g, \lambda) \mid \lambda \in P(C, c)\} \le \inf\{L(g, \lambda) \mid \lambda \in P(C) \;\wedge\; m(\lambda) \ge c\}$. Consequently, $e(c) = \inf\{L(g, \lambda) \mid \lambda \in P(C) \;\wedge\; m(\lambda) \ge c\}$.

If $c_1, c_2 \in C$ and $c_1 < c_2$, then $\{\lambda \in P(C) \mid m(\lambda) \ge c_2\} \subset \{\lambda \in P(C) \mid m(\lambda) \ge c_1\}$. Consequently, $e(c_2) = \inf\{L(g, \lambda) \mid \lambda \in P(C) \;\wedge\; m(\lambda) \ge c_2\} \ge \inf\{L(g, \lambda) \mid \lambda \in P(C) \;\wedge\; m(\lambda) \ge c_1\} = e(c_1)$.

(E) Copy the proof of Lemma 3.5(E).

(F) Copy the proof of Lemma 3.5(F).

(G) This follows from (C), (D), and (E). ■

Proof of Lemma 4.1 Consider B, g, η, and V *ex hypothesi*.

(A) $1 \in \arg\max_C V(B, .)$ if and only if $B - \eta(1) = V(B, 1) \ge V(B, c) = cB - \eta(c)$, i.e., $B \ge [\eta(1) - \eta(c)]/(1 - c)$, for every $c \in [0, 1)$.

(B) Using Lemma 3.6(B), $0 \in \arg\max_C V(B, .)$ if and only if $0 = V(B, 0) \ge V(B, c) = cB - \eta(c)$, i.e., $B \le \eta(c)/c$ for every $c \in (0, 1]$.

(C) If $c_0 > 0$ and $c \in [0, c_0)$, then Lemma 3.6(F) implies $V(B, c + \epsilon) = (c + \epsilon)B - \eta(c + \epsilon) = (c + \epsilon)B - \eta(c) > cB - \eta(c) = V(B, c)$ for $\epsilon \in (0, c_0 - c)$. So, $c \notin \arg\max_C V(B, .)$.

If η is continuous at 1, then Lemma 3.6(E) implies that η, and therefore $V(B, .)$, is continuous on C. As C is nonempty and compact, Weierstrass' theorem implies that $\arg\max_C V(B, .) \ne \emptyset$. As $[0, c_0) \cap \arg\max_C V(B, .) = \emptyset$ if $c_0 > 0$, we have $\arg\max_C V(B, .) \subset [c_0, 1]$.

If η is discontinuous at 1, then Lemma 3.6(D) implies that $\eta(1) - \eta(1-) = \epsilon > 0$. Consider $\delta \in (0, 1)$ such that $\delta B < \epsilon$. Then, $V(B, 1) - V(B, 1 - \delta) = B - (1 - \delta)B - [\eta(1) - \eta(1 - \delta)] \le \delta B - [\eta(1) - \eta(1-)] = \delta B - \epsilon < 0$. Hence, $1 \notin \arg\max_C V(B, .)$.

(D) If $\arg\max_C V(B, .) = \emptyset$, then it is trivially convex. Otherwise, suppose $c_1, c_2 \in \arg\max_C V(B, .)$. Let $t \in (0, 1)$. Since η is convex by Lemma 3.6(C), $V(B, .)$ is concave. Hence, $V(B, tc_1 + (1 - t)c_2) \ge tV(B, c_1) + (1 - t)V(B, c_2) \ge tV(B, c) + (1 - t)V(B, c) = V(B, c)$ for every $c \in C$. So, $tc_1 + (1 - t)c_2 \in \arg\max_C V(B, .)$.

In general, $\arg\max_C V(B, .)$ may be empty. Consider $e = 1_{\{1\}}$ on C and $B = 1/2$. As $V(1/2, .)$ is strictly increasing on $[0, 1)$, we have

$[0,1) \cap \arg\max_C V(1/2,.) = \emptyset$. Moreover, as $1/2 < 1 < 1/(1-c) = [\eta(1)-\eta(c)]/(1-c)$ for every $c \in [0,1)$, (B) implies $1 \notin \arg\max_C V(1/2,.)$. So, $e = 1_{\{1\}}$ implies $\arg\max_C V(1/2,.) = \emptyset$.

(E) By Lemma 3.6(C), $-V(B,.)$ is convex on $(0,1)$. By Lemma 7.23 in Aliprantis and Border [1], $-V(B,.)$ is subdifferentiable on $(0,1)$. Consider $c \in (0,1)$. Clearly, $c \in \arg\max_C V(B,.)$ if and only if $c \in \arg\min -V(B,.)$. By Lemma 7.10 in Aliprantis and Border [1], $c \in \arg\min -V(B,.)$ if and only if $0 \in -\partial V(B,c) = \partial\eta(c) - B$. Thus, $c \in \arg\max_C V(B,.)$ if and only if $B \in \partial\eta(c)$.

If η is differentiable at $c \in (0,1)$, then Theorem 7.25 in Aliprantis and Border [1] implies $\partial\eta(c) = \{D\eta(c)\}$.

(F) Consider $c = 1$. By (A), we have $\{B \in \Re_+ \mid 1 \in \arg\max_C V(B,.)\} = \cap_{c\in[0,1)}\{B \in \Re_+ \mid B \geq [\eta(1) - \eta(c)]/(1-c)\}$, which is a convex set.

Consider $c = 0$. By (B), we have $\{B \in \Re_+ \mid 0 \in \arg\max_C V(B,.)\} = \cap_{c\in(0,1]}\{B \in \Re_+ \mid B \leq \eta(c)/c\}$, which is a convex set.

Consider $c \in (0,1)$. Suppose $B_1, B_2 \in \{B \in \Re_+ \mid c \in \arg\max_C V(B,.)\}$, i.e., $c \in \arg\max_C V(B_1,.) \cap \arg\max_C V(B_2,.)$. By (E), $B_1, B_2 \in \partial\eta(c)$. Consider $t \in (0,1)$ and $c' \in (0,1)$. Then, $\eta(c')-\eta(c) \geq B_1(c'-c)$ and $\eta(c') - \eta(c) \geq B_2(c' - c)$. Since $\eta(c') - \eta(c) \geq tB_1(c' - c) + (1 - t)B_2(c' - c) = (tB_1 + (1 - t)B_2)(c' - c)$ for every $c' \in (0,1)$, we have $tB_1+(1-t)B_2 \in \partial\eta(c)$. By part (E), $c \in \arg\max_C V(tB_1+(1-t)B_2,.)$. So, $\{B \in \Re_+ \mid c \in \arg\max_C V(B,.)\}$ is a convex set.

By Lemma 3.6(E), there is a countable set E such that η, and therefore $V(B,.)$, is differentiable at every $c \in (0,1) \setminus E$. Consider $c \in (0,1) \setminus E$. Then, $D\eta(c) \in \Re_+$ and $B(c) := D\eta(c) \in \partial\eta(c)$. By (E), $c \in \arg\max_C V(B(c),.)$, and therefore, $B(c) \in \{B \in \Re_+ \mid c \in \arg\max_C V(B,.)\}$. Consider $B \in \Re_+$ such that $c \in \arg\max_C V(B,.)$. By (E), $B \in \partial\eta(c)$. As η is differentiable at c, we have $B = D\eta(c) = B(c)$. Hence, $\{B \in \Re_+ \mid c \in \arg\max_C V(B,.)\}$ is a singleton. ■

Proof of Lemma 5.5 Consider $\nu \in \Delta([0,\beta_1]) \setminus \Delta(b(C))$. Then, $\nu([0,\beta_1] \setminus b(C)) > 0$, i.e., $\nu(G^{-1}(\{c\}) \setminus \{b(c)\}) > 0$ for some $c \in \Phi(G)$. Hence, $a(c) = \max G^{-1}(\{c\}) > b(c)$ and $H_\nu(a(c)) - H_\nu(b(c)) = \nu((b(c), a(c)]) = \nu(G^{-1}(\{c\}) \setminus \{b(c)\}) > 0$. Define $t : [b(c), a(c)] \to \Re$ and $H_{\nu'} : \Re \to \Re$ by $t(x) = (a(c) - x)/(a(c) - b(c))$ and

$$H_{\nu'}(x) = \begin{cases} H_\nu(a(c)) + t(x)[H_\nu(x) - H_\nu(a(c))], & \text{if } x \in G^{-1}(\{c\}) \\ H_\nu(x), & \text{if } x \in \Re \setminus G^{-1}(\{c\}) \end{cases}$$

respectively. It follows that $H_{\nu'}(b(c)) = H_\nu(b(c))$, $H_{\nu'}(a(c)) = H_\nu(a(c))$, and $H_\nu < H_{\nu'}$ on $(b(c), a(c))$. It follows that $\int_{[x,\beta_1]} z\, dH_{\nu'}(z) = \beta_1 - xH_{\nu'}(x) -$

$\int_{[x,\beta_1]} dz\, H_{\nu'}(z) \le \beta_1 - xH_\nu(x) - \int_{[x,\beta_1]} dz\, H_\nu(z) = \int_{[x,\beta_1]} z\, dH_\nu(z)$ for every $x \in [0, \beta_1]$, with the inequality strict for $x \in (b(c), a(c))$, which implies $C_1(\nu') < C_1(\nu)$. Similarly, $C_2(\nu') = \int_{[0,\beta_1]} x[1 - H_{\nu'}(x)]\, dG(x) < \int_{[0,\beta_1]} x[1 - H_\nu(x)]\, dG(x) = C_2(\nu)$.

Clearly, $\int_{[0,\beta_1]}[H_{\nu'}(x) - H_\nu(x)]\, dG(x) = \int_{[0,b(c)]}[H_{\nu'}(x) - H_\nu(x)]\, dG(x) + \int_{(b(c),a(c))}[H_{\nu'}(x) - H_\nu(x)]\, dG(x) + \int_{[a(c),\beta_1]}[H_{\nu'}(x) - H_\nu(x)]\, dG(x) = 0$ as $H_{\nu'} = H_\nu$ on $[0, b(c)] \cup [\tau, \beta_1]$ and G is constant on $(b(c), a(c))$. So, $P(\nu') = \int_{[0,\beta_1]} G(x)\, dH_{\nu'}(x) = 1 - \int_{[0,\beta_1]} H_{\nu'}(x)\, dG(x) = 1 - \int_{[0,\beta_1]} H_\nu(x)\, dG(x) = \int_{[0,\beta_1]} G(x)\, dH_\nu(x) = P(\nu)$.

Therefore, $U_1(\nu') > U_1(\nu)$ and $U_2(\nu') > U_2(\nu)$. ■

Proof of Lemma 5.6 Consider G and Γ *ex hypothesi.*

(A) Consider $\lambda_1, \lambda_2 \in \Delta(C)$ with $\lambda_1 \circ b^{-1} = \Gamma(\lambda_1) = \Gamma(\lambda_2) = \lambda_2 \circ b^{-1}$.

Consider $c \in C$ and $x \in b^{-1} \circ G^{-1}([0, c])$. As b is strictly increasing by Lemma 3.1, $x = \bar{x}$. By Lemma 3.1, $x = \bar{x} = G \circ b(\bar{x}) = G \circ b(x) \in [0, c]$. So, $b^{-1} \circ G^{-1}([0, c]) \subset [0, c]$.

Consider $c \in C$ and $x \in [0, c]$. As $x \le c = \bar{c}$, Lemma 3.1 implies $G \circ b(x) \le G \circ b(c) = G \circ b(\bar{c}) = \bar{c} = c$. Hence, $x \in b^{-1} \circ G^{-1}([0, c])$. So, $b^{-1} \circ G^{-1}([0, c]) \supset [0, c]$.

As $b^{-1} \circ G^{-1}([0, c]) = [0, c]$ for every $c \in C$, we have $\lambda_1 = \lambda_1 \circ b^{-1} \circ G^{-1} = \lambda_2 \circ b^{-1} \circ G^{-1} = \lambda_2$. So, Γ is injective.

Consider $\nu \in \Delta(b(C))$. Since $b \circ G(x) = x$ for $x \in b(C)$, we have $\nu \circ G^{-1} \in \Delta(C)$ and $\Gamma(\nu \circ G^{-1}) = \nu \circ G^{-1} \circ b^{-1} = \nu \circ (b \circ G)^{-1} = \nu$. Thus, Γ is a surjection to $\Delta(b(C))$.

(B) Evidently, $b(C) \subset [0, \beta_1]$. As G is strictly increasing, Lemma 3.1 implies that b is continuous. As $b(0) = 0$, $b(1) = \beta_1$, and b is continuous, we have $b(C) \supset [0, \beta_1]$. Hence, $b(C) = [0, \beta_1]$ and $\Delta(b(C)) = \Delta([0, \beta_1])$. Using (A), Γ is a bijection from $\Delta(C)$ to $\Delta([0, \beta_1])$.

$\Delta(C)$ and $\Delta([0, \beta_1])$ are compact metric spaces (Parthasarathy [6], Theorem II.6.4). Consider a sequence $(\lambda_n) \subset \Delta(C)$ converging to $\lambda \in \Delta(C)$. If $h : [0, \beta_1] \to \Re$ is continuous, then the continuity of b implies $\int_{[0,\beta_1]} \lambda_n \circ b^{-1}(dx)\, h(x) = \int_C \lambda_n(dc)\, h \circ b(c) \to \int_C \lambda(dc)\, h \circ b(c) = \int_{[0,\beta_1]} \lambda \circ b^{-1}(dx)\, h(x)$. So, the sequence $(\lambda_n \circ b^{-1}) \subset \Delta([0, \beta_1])$ converges to $\lambda \circ b^{-1} \in \Delta([0, \beta_1])$. Hence, Γ is continuous.

Consider a closed set $E \subset \Delta(C)$. As $\Delta(C)$ is compact, E is compact. As Γ is continuous, $\Gamma(E)$ is compact. As $\Delta([0, \beta_1])$ is metric, it is Hausdorff, and therefore $\Gamma(E)$ is closed. Hence, Γ's function inverse is continuous. So, Γ is a homeomorphism. ■

Proof of Lemma 5.7 We start with a preliminary lemma:

Suppose $f : [a,b] \to \Re$ *is measurable. If* f *has a bounded derivative on* $[a,b] \setminus E$ *with* E *countable, then* f *has the Luzin property, i.e.,* $f(N) \in \mathcal{L}$ *and* $\text{Leb}(f(N)) = 0$ *for every* $N \subset [a,b]$ *such that* $N \in \mathcal{L}$ *and* $\text{Leb}(N) = 0$.

Proof. Consider f and E *ex hypothesi* and $N \subset [a,b]$ such that $N \in \mathcal{L}$ and $\text{Leb}(N) = 0$. Then, $N \cap E \in \mathcal{L}$, $N \setminus E \in \mathcal{L}$ and $\text{Leb}(N \setminus E) = 0$. As $N \cap E$ is countable, so is $f(N \cap E)$. Hence, $f(N \cap E) \in \mathcal{L}$ and $\text{Leb}(f(N \cap E)) = 0$.

As f is differentiable on $N \setminus E$, $\text{Leb}^*(f(N \setminus E)) \leq \int_{N \setminus E} \text{Leb}(dx)\, |Df(x)|$ (Bruckner et al. [2], Lemma 7.13). Since Df is bounded on $N \setminus E$ and $\text{Leb}(N \setminus E) = 0$, we have $\text{Leb}^*(f(N \setminus E)) = 0$.

Consider $T \in 2^{\Re}$. As Leb^* is subadditive, $\text{Leb}^*(T \cap f(N \setminus E)) + \text{Leb}^*(T \setminus f(N \setminus E)) \geq \text{Leb}^*(T)$. As $\text{Leb}^*(T \cap f(N \setminus E)) \leq \text{Leb}^*(f(N \setminus E)) = 0$ and $\text{Leb}^*(T \setminus f(N \setminus E)) \leq \text{Leb}^*(T)$, we have $\text{Leb}^*(T \cap f(N \setminus E)) + \text{Leb}^*(T \setminus f(N \setminus E)) = \text{Leb}^*(T)$. Consequently, $f(N \setminus E) \in \mathcal{L}$ by the Caratheodory characterisation of $\mathcal{L}$ and $\text{Leb}(f(N \setminus E)) = \text{Leb}^*(f(N \setminus E)) = 0$.

Hence, $f(N) = f(N \cap E) \cup f(N \setminus E) \in \mathcal{L}$ and $\text{Leb}(f(N)) \leq \text{Leb}(f(N \cap E)) + \text{Leb}(f(N \setminus E)) = 0$. So, $\text{Leb}(f(N)) = 0$. ■

Consider G and λ *ex hypothesi.*

1. By hypothesis, G is continuous and therefore Borel measurable. As it is increasing, it has bounded variation. Consider $N \in \mathcal{L}$ such that $\text{Leb}(N) = 0$.

   Then, $N \setminus [0, \beta_1] \in \mathcal{L}$ and $\text{Leb}(N \setminus [0, \beta_1]) = 0$. As $G(N \setminus [0, \beta_1]) \subset \{0, 1\}$, we have $G(N \setminus [0, \beta_1]) \in \mathcal{L}$ and $\text{Leb}(G(N \setminus [0, \beta_1])) = 0$.

   Also, $N \cap [0, \beta_1] \in \mathcal{L}$ and $\text{Leb}(N \cap [0, \beta_1]) = 0$. Using (b) and the preliminary lemma, $G(N \cap [0, \beta_1]) \in \mathcal{L}$ and $\text{Leb}(G(N \cap [0, \beta_1])) = 0$.

   So, $G(N) = G(N \setminus [0, \beta_1]) \cup G(N \cap [0, \beta_1]) \in \mathcal{L}$ and $0 \leq \text{Leb}(G(N)) \leq \text{Leb}(G(N \setminus [0, \beta_1])) + \text{Leb}(G(N \cap [0, \beta_1])) = 0$. Hence, G satisfies the Luzin property. The Banach-Zarecki theorem (Bruckner et al. [2], Theorem 7.14) implies that G is absolutely continuous.

2. F_λ is absolutely continuous by an analogous argument.

3. Let $B_0 = b(C)$. Since B_0 is a countable union of disjoint intervals, $B_0 \in \mathcal{B}([0, \beta_1])$.

   Since G is continuous, $\bar{c} = c$ for every $c \in C$. Consider $b_1, b_2 \in B_0$ such that $b_1 < b_2$. Then, there exist $c_1, c_2 \in C$ such that $b(c_1) = b_1 < b_2 = b(c_2)$, i.e., $c_1 < c_2$. Using Lemma 3.1, $G(b_1) = G \circ b(c_1) = c_1 < c_2 = G \circ b(c_2) = G(b_2)$. Therefore, G is strictly increasing on B_0. So, G's restriction to B_0 is continuous and injective, with function inverse b.

   As F_λ is absolutely continuous by step 2, so is λ (Bruckner et al. [2], Theorem 5.28). Consider $E \in \mathcal{B}([0, \beta_1])$ such that $\text{Leb}(E) = 0$. As $B_0 \in \mathcal{B}([0, \beta_1])$, we have $E \cap B_0 \in \mathcal{B}([0, \beta_1])$ and $\text{Leb}(E \cap B_0) = 0$. As

G satisfies the Luzin property (see step 1), $\mathrm{Leb}(G(E \cap B_0)) = 0$. As λ is absolutely continuous, $\Gamma(\lambda)(E) = \lambda \circ b^{-1}(E) = \lambda \circ b^{-1}(E \cap B_0) = \lambda(G(E \cap B_0)) = 0$. So, $\Gamma(\lambda)$ is absolutely continuous.

If $x \in [0, \beta_1]$, then Lemma 3.1 and the continuity of G imply $H_{\Gamma(\lambda)}(x) = \Gamma(\lambda)([0, x]) = \lambda \circ b^{-1}([0, x]) = \lambda(G([0, x])) = \lambda([0, G(x)]) = F_\lambda \circ G(x)$.

As G and F_λ are continuous, so is $H_{\Gamma(\lambda)}$. As $\Gamma(\lambda)$ is absolutely continuous and $H_{\Gamma(\lambda)}$ is increasing and continuous, $H_{\Gamma(\lambda)}$ is absolutely continuous (Bruckner et al. [2], Theorem 5.28).

4. As B_0 is measurable, so is $B_1 := [0, \beta_1] \setminus B_0 = \cup_{c \in \Phi(G)} [G^{-1}(\{c\}) \setminus \{b(c)\}]$. Clearly, $b^{-1}(B_1) = \emptyset$ and $A := \{\max G^{-1}(\{c\}) \mid c \in \Phi(G)\} \cup b(\Phi(G))$ is countable.

   Consider $x \in B_1 \setminus A$. Then, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset G^{-1}(\{G(x)\})$. Hence, $DG = 0$ on $B_1 \setminus A$ and therefore DG is continuous on $B_1 \setminus A$. Since $b^{-1}(B_1) = \emptyset$, we have $\Gamma(\lambda)(B_1) = \lambda \circ b^{-1}(B_1) = 0$, and therefore $H_{\Gamma(\lambda)} = H_{\Gamma(\lambda)}(b(c))$ on $G^{-1}(\{c\})$ for every $c \in \Phi(G)$. Hence, $H_{\Gamma(\lambda)}$ is differentiable on $B_1 \setminus A$ and $DH_{\Gamma(\lambda)} = 0$ on $B_1 \setminus A$. So, $DH_{\Gamma(\lambda)}$ exists and is continuous and bounded on the co-countable subset $B_1 \setminus A$ of B_1.

   As G is strictly increasing on B_0, $B_0 \cap G^{-1}(L)$ is countable. Hence, $B_0 \cap [M \cup G^{-1}(L)]$ is countable. If $x \in B_0 \setminus [M \cup G^{-1}(L)]$, then G is differentiable at x, $G(x) \in C \setminus L$, and F_λ is differentiable at $G(x)$. So, $H_{\Gamma(\lambda)} = F_\lambda \circ G$ is differentiable on $B_0 \setminus [M \cup G^{-1}(L)]$ with $DH_{\Gamma(\lambda)}(.) = DF_\lambda(G(.))DG(.)$. Using (b), $DH_{\Gamma(\lambda)}$ is defined, continuous, and bounded on the co-countable subset $B_0 \setminus [M \cup G^{-1}(L)]$ of B_0.

   So, $DH_{\Gamma(\lambda)}$ is defined, continuous, and bounded on $B_0 \setminus [M \cup G^{-1}(L)] \cup B_1 \setminus A \subset [0, \beta_1]$, which is a co-countable subset of $[0, \beta_1]$.

5. As $H_{\Gamma(\lambda)}$ is continuous, so is $h_{\Gamma(\lambda)}$. As $H_{\Gamma(\lambda)}$ increasing, so is $h_{\Gamma(\lambda)}$; therefore, $h_{\Gamma(\lambda)}$ has bounded variation. Using step 4, $h_{\Gamma(\lambda)}$ is differentiable on the co-countable subset $B_0 \setminus [M \cup G^{-1}(L)] \cup B_1 \setminus A$ of $[0, \beta_1]$ and $Dh_{\Gamma(\lambda)}$ is bounded on this set. The preliminary lemma implies that $h_{\Gamma(\lambda)}$ satisfies the Luzin property. Hence, $h_{\Gamma(\lambda)}$ is absolutely continuous by the Banach-Zarecki theorem (Bruckner et al. [2], Theorem 7.14). ∎

Proof of Lemma 5.8 Consider $\lambda \in \Delta(C)$ and suppose the hypotheses of Lemma 5.7 are satisfied.

(A) By (19), $C_1(\Gamma(\lambda)) = \int_{[0,\beta_1]} [\int_{[x,\beta_1]} z \, dH_{\Gamma(\lambda)}(z)] \, dG(x)$. Since $H_{\Gamma(\lambda)}$ is continuous by Lemma 5.7, we have $\int_{[x,\beta_1]} z \, dH_{\Gamma(\lambda)}(z) = \beta_1 - h_{\Gamma(\lambda)}(x) -$

$\int_{[x,\beta_1]} dz\, H_{\Gamma(\lambda)}(z)$. Therefore, $C_1(\Gamma(\lambda)) = \beta_1 - \int_{[0,\beta_1]} h_{\Gamma(\lambda)}(x)\, dG(x) - \int_{[0,\beta_1]}[\int_{[x,\beta_1]} dz\, H_{\Gamma(\lambda)}(z)]\, dG(x)$.

It follows from Lemma 5.7 that G and $h_{\Gamma(\lambda)}$ are absolutely continuous. Applying Theorem 7.32 in Wheeden and Zygmund [7], we have $\int_{[0,\beta_1]} h_{\Gamma(\lambda)}(x)\, dG(x) = \beta_1 - \int_{[0,\beta_1]} G(x)\, dh_{\Gamma(\lambda)}(x)$ and

$$\begin{aligned}\int_{[0,\beta_1]} G(x)\, dh_{\Gamma(\lambda)}(x) &= \int_{[0,\beta_1]} \mathrm{Leb}(dx)\, G(x) D h_{\Gamma(\lambda)}(x) \\ &= \int_{[0,\beta_1]} \mathrm{Leb}(dx)\, G(x)\big[x D H_{\Gamma(\lambda)}(x) + H_{\Gamma(\lambda)}(x)\big]\end{aligned}$$

Since G and $H_{\Gamma(\lambda)}$ are continuous, $\int_{[0,\beta_1]}[\int_{[x,\beta_1]} dz\, H_{\Gamma(\lambda)}(z)]\, dG(x) = \int_{[0,\beta_1]} dx\, G(x) H_{\Gamma(\lambda)}(x) = \int_{[0,\beta_1]} \mathrm{Leb}(dx)\, G(x) H_{\Gamma(\lambda)}(x)$ by Lebesgue's theorem (Bruckner et al. [2], Theorem 5.20).

Combining terms and then applying Theorem 7.32 in Wheeden and Zygmund [7], we have $C_1(\Gamma(\lambda)) = \int_{[0,\beta_1]} \mathrm{Leb}(dx)\, G(x) x D H_{\Gamma(\lambda)}(x) = \int_{[0,\beta_1]} G(x) x\, dH_{\Gamma(\lambda)}(x)$ as $H_{\Gamma(\lambda)}$ is absolutely continuous by Lemma 5.7 and $x \mapsto xG(x)$ is continuous. It follows from (15) that $C_1(\Gamma(\lambda)) = \int_{[0,\beta_1]} \Gamma(\lambda)(dx)\, G(x) x = \int_{[0,\beta_1]} \lambda \circ b^{-1}(dx)\, G(x) x = \int_C \lambda(dc)\ b(c) G \circ b(c) = \hat{C}_1(\lambda)$.

Consider the second-price auction. Using (16) and changing variables, $\hat{C}_2(\lambda) = \int_C \lambda(dc) \int_{[0,b(c)]} x\, dG(x) = \int_{[0,\beta_1]} \Gamma(\lambda)(dy) \int_{[0,y]} x\, dG(x) = \int_{[0,\beta_1]}[\int_{[0,y]} x\, dG(x)]\, dH_{\Gamma(\lambda)}(y)$. Integrating by parts and copying the above arguments, $\hat{C}_2(\lambda) = \int_{[0,\beta_1]} y\, dG(y) - \int_{[0,\beta_1]} H_{\Gamma(\lambda)}(y) y\, dG(y) = \int_{[0,\beta_1]}[1 - H_{\Gamma(\lambda)}(y)] y\, dG(y) = C_2(\Gamma(\lambda))$ by (20).

(B) By Lemma 3.1, as G is continuous, b is strictly increasing, $c = \bar{c}$ for every $c \in C$, and $G \circ b(c) = c$ for every $c \in C$. Using (14) and (18), $m(\lambda) = \int_C \lambda(dc)\, c = \int_C \lambda(dc)\, G \circ b(c) = \int_{[0,\beta_1]} \lambda \circ b^{-1}(dx)\, G(x) = \int_{[0,\beta_1]} \Gamma(\lambda)(dx)\, G(x) = P(\Gamma(\lambda))$.

(C) follows from (A), (B), (14), and (21). ■

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