DYNAMIC EFFICIENCY IN A TWO-SECTOR OVERLAPPING GENERATIONS MODEL

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Dynamic Efficiency in a Two-Sector Overlapping Generations Model

Partha Sen
Delhi School of Economics

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ABSTRACT

This paper looks at the conditions under which we may have welfare improving capital accumulation in two-sector two-period overlapping generations models. It is found that both the usual conditions of the rate of interest exceeding the population growth rate and profits exceeding investment may give misleading answers. Finally there is also the possibility of asset bubbles even with dynamic efficiency.

1. INTRODUCTION

It is well known that in two period overlapping generations models with capital, a competitive market economy may accumulate "too much" capital. Capital in these models is the sole store of value and households accumulate it to finance consumption in their old age. This drives down the rate of return on capital in its role as a factor of production[1].

The literature on overaccumulation of capital is over four decades old (see Malinvaud (1953), Cass (1972) for example). In the overlapping generations framework, Diamond (1965) discusses the conditions under which we may say that the competitive economy has accumulated too much capital (also see Buiter (1981)). In a more recent work Abel, Mankiw, Summers and Zeckhaüser (1989) generalize Diamond's result and test for "dynamic efficiency" of (i.e. the absence of overaccumulation in) the US economy.

Both the Diamond and the Abel et.al. studies use the concept of the Golden Rule of Accumulation due to Phelps (1961). Diamond shows that whenever the steady state rate of interest exceeds the (exogenous) growth rate of population, the economy is dynamically efficient in the steady state. Abel et.al look outside the steady state and propose a criterion (henceforth referred to as the AMSZ criterion) which says that if investment in an economy is less than the profits in the economy then we have dynamic efficiency. In the steady state, (and certainty) their criterion is identical to Diamond's.

Both these criteria make intuitive sense. If the interest rate is less than the population growth rate, the net rate of return to capital for the society is negative. Similarly, if investment is greater than profits, society is putting more into capital accumulation than it is getting in return. Social welfare

[1] This is due to the "double infinity" property of these models. See Shell (1971) for a discussion.

could be increased by consuming some of the capital. This would raise the welfare of the individuals consuming the additional amount and raise the rate of return for every one in the economy.

These issues are important because many government policies (e.g., debt policy) could potentially increase welfare if the economy is overaccumulating capital. In such an economy asset bubbles could also exist. The issue of Ricardian Equivalence also hinges on dynamic efficiency[2]. Thus the issue of dynamic efficiency is more than a theoretical curiosum.

In this paper I ask whether these criteria carry over to two sector models. I discuss an example and show that they do not. The reason is not far to seek. The one sector world is not contaminated by such things as relative factor intensities and relative prices. In a two sector model there is a difference between the consumption wage and the product wage, the own rate of interest and the consumption rate of interest. We find that an economy could be dynamically efficient but the Diamond and the AMSZ criteria fail to detect it. Moreover, a dynamically efficient economy may allow asset bubbles to exist. All hell breaks loose[3].

2. THE MODEL

The economy consists of overlapping generations of individuals or households. Each household lives for two periods. It supplies one unit of labour in the first period of its life and in the second period consumes the saving from the first

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[2] See Tirole (1985) and Weil (1987a) on asset bubbles. Weil (1987b) discusses bequests and dynamic efficiency.

[3] There are many examples of two sector models in a two period overlapping generations framework. In an open economy context see e.g., Persson and Svensson (1985) and Matsuyama (1988). Galor (1992) discusses issues relating to the existence of equilibrium in such models. For a small open economy the rate of return on capital is exogenously given and so the issue of overaccumulation does not arise.

period plus the return on these savings. There are no bequests or inheritances. The population is growing at a constant rate.

The representative household born in time period t maximizes the following utility function

$$U_t = \log C_t^1 + (1+\delta)^{-1} \log C_t^2 \tag{1}$$

where C_t^i is the consumption in period i of a household born in t and $\delta > 0$ is the rate of time preference.

Its lifetime constraint is

$$W_t = C_t^1 + (1+r_{t+1})^{-1} . C_t^2 \tag{2}$$

where W_t is the wage rate in time period t and r_{t+1} the own interest rate on one period consumption loans between t and t + 1.

This yields

$$C_t^1 = [(1+\delta)/(2+\delta)]W_t$$

and $C_t^2 = [(1+r_{t+1}) / (2+\delta)]W_t$

The indirect utility function is given by

$$V_t = m + [(2+\delta)\log W_t + \log(1+r_{t+1})]/(1+\delta) \tag{3}$$

where m is a constant.

The production side of the economy is represented by the two cost-equal-to-price equations. The consumption good (C) and the investment good (I) are produced under conditions of constant returns to scale using the two inputs, capital (K) and labour (L). All inputs are mobile between sectors instantaneously.

Capital is assumed to depreciate completely in the process of production -- not a bad assumption for a model where a single period corresponds to about 35 to 40 calender years[4].

$$a_{LC} \cdot W_t + a_{KC} \cdot R_t = 1 \tag{4}$$

$$a_{LI} \cdot W_t + a_{KI} \cdot R_t = p_t \tag{5}$$

where a_{ij} is the requirement of the i^{th} input (i = K, L) in the production of the j^{th} good (j = C, I), and p is the relative price of the investment good in terms of the numeraire good C. R is the gross return on capital. Since we assume capital depreciates completely in the process of production, we have in equilibrium $(1 + r_{t+1}) = R_{t+1} / p_t$.

There are two goods markets and two factor markets. By Walras Law if three of these are in equilibrium in any period then so is the fourth one. We thus have

$$a_{LC} \cdot C_t + a_{LI} \cdot I_t = 1 \tag{6a}$$

$$a_{KC} \cdot C_t + a_{KI} \cdot I_t = k_t \tag{6b}$$

$$p_t \cdot I_t = S_t = (2 + \delta)^{-1} W_t \tag{6c}$$

Equations (6a), (6b) and (6c) are the market clearing conditions for labour, capital and investment goods markets respectively. The variable C_t is the production per worker of the consumption good, I_t is the output per worker of the investment good, S_t is the saving per head of the young in period t and k_t is the capital stock per worker (all in time period t). Given the logarithmic form of the utility function we have a proportion $1/(2+\delta)$ of labour income saved by the young in (6c).

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[4] This is for analytical convenience only. Dropping this would make $1 + r_{t+1} = (p_{t+1} + R_{t+1}) / p_t$. The presence of the capital gains term p_{t+1}/p_t makes no difference to our results since our discussion is limited to the steady state. It affects the dynamics of the economy, though.

Finally, the dynamics of the economy is represented by

$$(1+n)k_{t+1} = I_t \tag{7}$$

Note that (4) and (5) yield

$$\theta_{LC}.\hat{W}_t + \theta_{KC}.\hat{R}_t = 0 \tag{8a}$$

$$\theta_{LI}.\hat{W}_t + \theta_{KI}.\hat{R}_t = \hat{p}_t \tag{8b}$$

where θ_{ij} is the share of the i^{th} input in the j^{th} sector price and a hat over a variable denotes a percentage change.

From (8a) and (8b) we have

$$\hat{W}_t/\hat{p}_t = -\theta_{KC}/\Delta \tag{9a}$$

$$\hat{R}_t/\hat{p}_t = \theta_{LC}/\Delta \tag{9b}$$

where $\Delta \equiv \theta_{LC} - \theta_{LI} = \theta_{KI} - \theta_{KC}$

We assume technology in both sectors is Cobb-Douglas i.e., $\sigma_c = \sigma_I = 1$ where σ_j is the elasticity of substitution between inputs in the j^{th} industry. This implies that θ_{ij}'s are constant.

From (6a), (6b) and (6c) we then have

$$\lambda_{LC}.\hat{C}_t + \lambda_{LI}.\hat{I}_t = [\hat{W}_t - \hat{R}_t](\lambda_{LC}.\theta_{KC} + \lambda_{LI}.\theta_{KI}) \tag{10a}$$

$$\lambda_{KC}.\hat{C}_t + \lambda_{KI}.\hat{I}_t = [\hat{W}_t - \hat{R}_t](\lambda_{KC}.\theta_{LC} + \lambda_{KI}.\theta_{LI}) + \hat{k}_t \tag{10b}$$

$$\hat{p}_t + \hat{I}_t = \hat{W}_t \tag{10c}$$

where λ_{ij} is the share of the j^{th} sector in the total employment of the i^{th} input

If we substitute for $\hat{W}_t$ and $\hat{R}_t$ from (9a) and (9b) into (10a), (10b) and (10c), we can solve for $\hat{C}_t$, $\hat{I}_t$ and $\hat{p}_t$ in terms of $\hat{k}_t$.

Equation (7) can be linearized around the steady state and written as

$$d k_{t+1} = (\hat{I}_t / \hat{k}_t) . dk_t \tag{11}$$

where $dk_{t+i} \equiv k_{t+i} - k^*$ is the deviation of the t+i period capital per worker from its steady state value k^* (an asterisk denotes a steady state value).

From (10a), (10b), (10c) and (11) we find that the long - run equilibrium is stable and the adjustment is monotonic (the details are in Appendix A).

$$d\ k_{t+1} = \theta_{KI}\ .\ dk_t \tag{12}$$

From (10a), (10 b) and (10c) we also have

$$\hat{p}_t = -\Delta \hat{k}_t \tag{13}$$

where Δ is defined below equation (9 b).

Next substituting (9a), (9b) and (13) into (4) we have in the steady state

$$dV = [(2+\delta)\theta_{KC} - \theta_{LI}]\ (1+\delta)^{-1} \hat{k}^* \tag{14}$$

Note that in this model $\hat{W} / \hat{k}$ and $(\hat{R} - \hat{p}) / \hat{k}$ do not depend on relative capital intensities of the two sectors[5].

[5] This as well as the stability being independent of the relative capital intensities is a feature due to the fact that all elasticities of substitution are assumed to be unity. Azariades (1993) discusses this example (p. 264)

Capital accumulation raises steady state welfare iff
$$(2+\delta)\,\theta_{KC} > \theta_{LI} \tag{15}$$

If $\theta_{LI} > (2+\delta)\,\theta_{KC}$, we have capital overaccumulation and a reduction of capital will raise welfare in the new steady state.

An important point which should be remembered is that with Cobb-Douglas technology θ_{ij}'s are constants and so the sign of $dV/dk*$ depends only on technology and preferences (through the rate of time preference). And whether $dV/dk*$ is positive or negative depends on "knife-edge" values of θ_{LI} and θ_{KC} (given the taste parameter δ).

But what does this have to do with the interest rate exceeding or falling short of the (exogenous) growth rate of population or the profits exceeding investment ? First it should be obvious that in the steady state the Diamond and the AMSZ criteria are identical in this model. To see this note that we have

$$R^{*}/p^{*} > (1+n)$$

or $$R^{*}k^{*} > (1+n)\,.\,p^{*}k^{*}$$

$$R^{*}k^{*} > p^{*}I^{*} \tag{16}$$

Now let us go back to the question posed at the beginning of the last paragraph. The answer is: "Not a whole lot". In particular, $dV/dk*$ can be negative with $r* > n$ where $r^{*} = (R^{*}/p^{*}) - 1$. Of course, $dV/dk^{*} > 0$ is also possible with $r^{*} > n$. I discuss this in detail in the next section.

The AMSZ specification also does not lead us very far. In our example we then require (equation (16) above)

$$p^{*}.I^{*} < R^{*}.k^{*}$$

or $\quad W^{*}.a_{LI}.I^{*} < R^{*}.a_{KC}.C^{*}$

or $\quad \theta_{LI} < \theta_{KC}.C^{*}/p^{*}I^{*}$ (17)

Now in the steady state the saving equal to investment equation is

$$p^{*}I^{*} = (2+\delta)^{-1}W^{*} \tag{18}$$

and

$$C^{*} = [(1+\delta)W^{*} + R^{*}W^{*}/p^{*}(1+n)]/(2+\delta) \tag{19}$$

In (19) the first term within the square brackets is the expenditure of the young and the second term the expenditure of the old. There are $L_{t-1} = L_t/(1+n)$ of the latter in time period t where L_{t+i} is the work-force (the young) at time t + i,

Substituting (19) in (17) the AMSZ criterion reduces to

$$\theta_{LI} < \theta_{KC}.(1+\delta+R^{*}/p^{*}(1+n)) \tag{20}$$

We see that (20) reduces to (15) only when we have R^{*}/p^{*} = (1 + n) i.e., we are at the so called "Golden Rule" value of the capital stock.

Dynamic inefficiency according to AMSZ and Diamond occurs when the inequality in (20) is reversed i.e,

$$(1+\delta+R^{*}/(1+n)p^{*})\ \theta_{KC} < \theta_{LI} \tag{21}$$

From (20) and (15) it is quite possible that we have capital overaccumulation i.e., $\theta_{LI} > (2+\delta)\theta_{KC}$ and R^{*}/p^{*} > (1 + n). We could then have

$$(2+\delta)\theta_{KC} < \theta_{LI} < (1+\delta+R^{*}/p^{*}(1+n))\ \theta_{KC} \tag{22}$$

Similarly if $R^{*}/p^{*} < (1+n)$ we could have

$$(2+\delta)\,\theta_{KC} > \theta_{LI} > (1+\delta+R^{*}/p^{*}(1+n))\,\theta_{KC} \tag{23}$$

In equation (22) capital accumulation is immiserizing but the AMSZ criterion and the Diamond criterion are not violated. In (23), <u>per contra</u> , capital accumulation increases welfare but AMSZ and Diamond would pronounce the system guilty of capital overaccumulation.

In Figure 1 we have k^{*} on the horizontal axis and $(2+\delta)\theta_{KC}, \theta_{LI}$ and $(1+\delta+R^{*}/p^{*}(1+n))$ on the vertical axis. $(1+\delta+R^{*}/p^{*}(1+n))$ and k^{*} are negatively related. k_G^{*} is the Golden Rule value of k^{*}. At k_A^{*} we have the inequalities in (22) being satisfied. In Figure 2 at k_B^{*} we have (23) holding.

A very important implication of this example is that there is no longer a one-to-one relationship between dynamic inefficiency and the existence of speculative bubbles. For example, if inequality (23) is satisfied then we could have a bubble asset co-existing with dynamic efficiency. The economy grows at a rate n while the rate of return to capital is (R^{*}/p^{*}) (and this is the rate at which the bubble asset grows). With (R^{*}/p^{*}) < (1 + n) the bubble does not get to dominate the economy, while $(2+\delta)\,\theta_{kc} > \theta_{LI}$ implies dynamic efficiency. An observer just looking at the Diamond and the AMSZ criteria would however (erroneously) conclude that the bubble exists because the economy is dynamically inefficient.

3. DISCUSSION AND CONCLUSIONS

Why does the two sector economy considered in the paper not reproduce the conditions for dynamic efficiency of the one sector model? The model we have considered is the simplest in the two sector family where all elasticities of substitution are unity - both of the intertemporal consumption and the atemporal production varieties.

To get a handle on this issue let us start with how we perform welfare calculation in the one sector model. The direct utility function in the steady state is

$$U(C^1, C^2) \tag{24}$$

and the indirect utility function

$$V(W^*, 1+r^*) \tag{25}$$

Differentiation of (25) reveals

$$dV = V_1 dW^* + V_2 dr^* \tag{26}$$

Now use the knowledge that

$$V_1 = U_1,\ V_2 = U_2 S \text{ and } U_1 = U_2(1+r)$$

$$dV/V_1 = dW^* + (S^*/(1+r^*))\,dr^* \tag{27}$$

Finally, substitute from the capital accumulation equation and the factor price frontier i.e.,

$$dW^*/dr^* = -k^* \text{ and } k^*(1+n) = S^* \tag{28}$$

$$dV/V_1 = dW^*(r^*-n)/(1+r^*) \tag{29}$$

We know W^* is an increasing function of k^*. So whether an increase in steady state k increases welfare depends on r^*-n.

In a two sector framework there are at least three changes which need to be made. First, $(1+r^*)$ is replaced by something akin to R^*/p^* of this paper or alternatively, if I is the numeraire the consumption wage will differ from the product wage rate. Second, in the accumulation equation S^* is in terms of the consumption good while k^* is in I units requiring the presence of the relative price. Finally, the factor price relation is obtained from the consumption good, sector viz. $dW^*/dR^* = -a_{KL}/a_{LC}$ i.e., not the capital intensity of economy as a whole but of the numeraire good sector alone.

Unlike in equation (25) we have the consumption rate of interest R^*/p^* in the indirect utility function. Proceeding as we did above we get an expression.

$$dV/V_1 = [1-(1+n)k^*.(R^*/p^*)^{-1}(\frac{dR^*}{dW^*} - \frac{R^*}{p^*}.\frac{dp^*}{dW^*})]dW^* \qquad (30)$$

Given that the term $dR^*/dW^* = -1/k_c^*$ where $k_c^* \equiv a_{KC}/a_{LC}$, the ratio k^*/k_c^* is unity only in a one sector framework. In general, it depends on sectoral capital intensities. The relative price term dp^*/dW^* also depends on capital intensities. Finally dW^*/dk^* also depends on sectoral capital intensities.

This is as far as we can go with the general model. What about the model described in this paper? Note (30) can be rewritten as

$$dV/V_1 = dW^*[1-(\frac{S^*}{R^*/p^*})(\frac{dR^*/p^*}{dW^*})] \qquad (31)$$

The expression in the square brackets is equal to

$$1-(S^*/W^*)(\hat{R}^*/\hat{W}^* - \hat{p}^*/\hat{W}^*) \qquad (32)$$

Now $S^*/W^* = \frac{(1+n)}{R^*p^*}\frac{R^*k^*}{W^*}$

But given our logarithmic preferences

$$S^*/W^* = \frac{1}{2+\delta}$$

Thus the effect of a reduction of R^*/p^* as capital accumulation proceeds is exactly offset by a rise in R^*k^*/W^*.

Finally $(\hat{R}^* - \hat{p}^*)/\hat{W}^* = -\theta_{LI}/\theta_{KC}$. So no matter what the relation between (1+n) and R^*/p^*, we have dynamic efficiency because the expression in (15) is positive.

Another way of looking at the problem is to remember that in our model capital accumulation increases steady-state welfare iff (equation (15) above)

$$(2+\delta)\,\theta_{KC} > \theta_{LI}$$

or $$\theta_{KC} > \frac{1}{2+\delta}\cdot\theta_{LI}$$

or $$\frac{R^*}{p^*} > \frac{1}{2+\delta}\,W^*\,\frac{\theta_{LI}}{\theta_{KC}}\,\frac{R^*}{p^*W^*}$$

or, finally $$\frac{R^*}{p^*} > (1+n)\,\frac{\theta_{LI}}{\theta_{KC}}\,\frac{R^*K^*}{W^*} \tag{33}$$

Now the term after (1 + n) in equation (33) is in general not equal to unity except in a one-sector model.

In Figure 3, I show the steady state behaviour of my model using the diagrammatic apparatus due to Matsuyama (1991) (the details are in Appendix B).

In conclusion, we can say that the welfare properties of the two-sector overlapping generations model studied in this paper are very different from those of the one sector models. There is no simple relationship between dynamic efficiency on the one hand and the rate of interest exceeding the population growth rate or profits exceeding investment on the other hand. A dynamically efficient economy could peacefully coexist with asset bubbles.

APPENDIX A

We derive the conditions for the stability of the model. First, savings equal to investment from (10c) gives

$$(2+\delta)^{-1}.W_t = p_t.I_t$$

or $\quad \hat{W}_t = \hat{p}_t + \hat{I}_t$

or $\quad \hat{I}_t = -\theta_{KI}.\Delta^{-1}\hat{p}_t \qquad$ (A.1)

Using (A.1), (9a), (9b), (10a) and (10b) we get

$$\lambda_{LC}\hat{C}_t + \lambda_{LC}\,\theta_{KC}.\Delta^{-1}.\hat{p}_t = 0 \qquad \text{(A.2a)}$$

$$\lambda_{KC}\hat{C}_t - (\lambda_{KC}\theta_{LC}+\lambda_{KI}).\Delta^{-1}.\hat{p}_t = \hat{k}_t \qquad \text{(A.2b)}$$

From (A.2a) and (A.2b) we get

$$\hat{p}_t/\hat{k}_t = -\Delta$$

and hence

$$\hat{I}_t\,/\,\hat{k}_t = \theta_{KI}$$

Finally

$$(1+n).k_{t+1} = I_t$$

so $\quad d\,k_{t+1} = \theta_{KI}.dk_t$ around the steady state.

APPENDIX B

In Figure 3, I portray the steady state of the economy in the dynamically efficient case using the diagrammatic apparatus of Matsuyama (1991).

The axes have log R^* and log W^* respectively. The factor price relationship

$$(d\ln W^*/d\ln R^*)_{FF} = -\theta_{KC}/\theta_{LC}$$

The slope of dV is

$$(d\ln W/d\ln R^*)_{VV} = (-1/(2+\delta))\,(\theta_{LI}/\theta_{LC})$$

Finally the steady state relationship between k and S

$$(1+n)k^* = (2+\delta)^{-1}.W^*$$

or $$(d\ln W^*/d\ln R^*)_{DD} = -(1/\theta_{LC})$$

So as in Matsuyama (1991), FF is flatter than DD. VV is flatter than FF (for dynamic efficiency) if

$$(2+\delta)\theta_{KC} > \theta_{LI}$$

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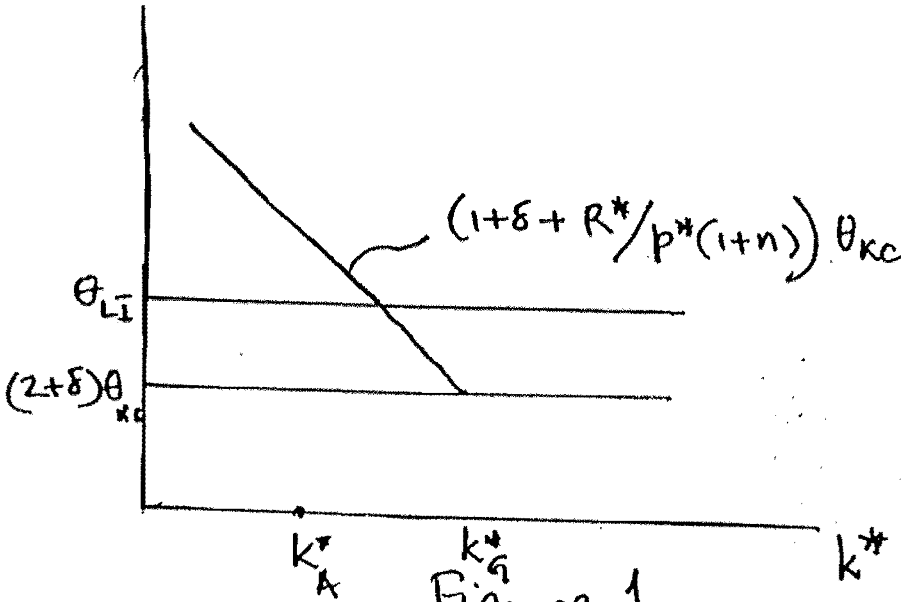

Figure 1.

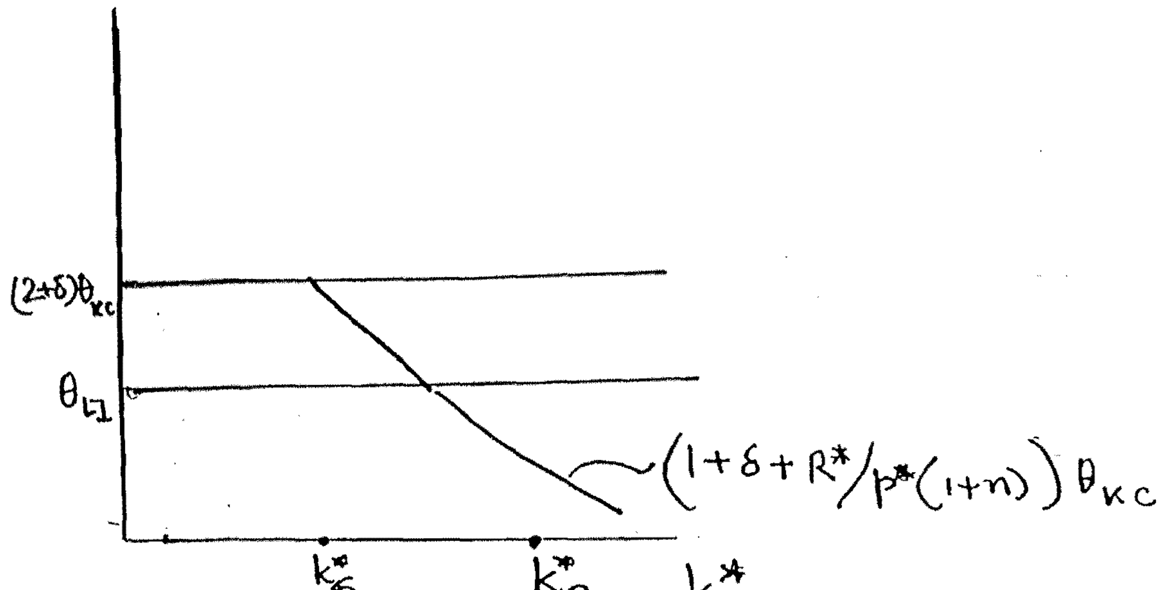

Figure 2.

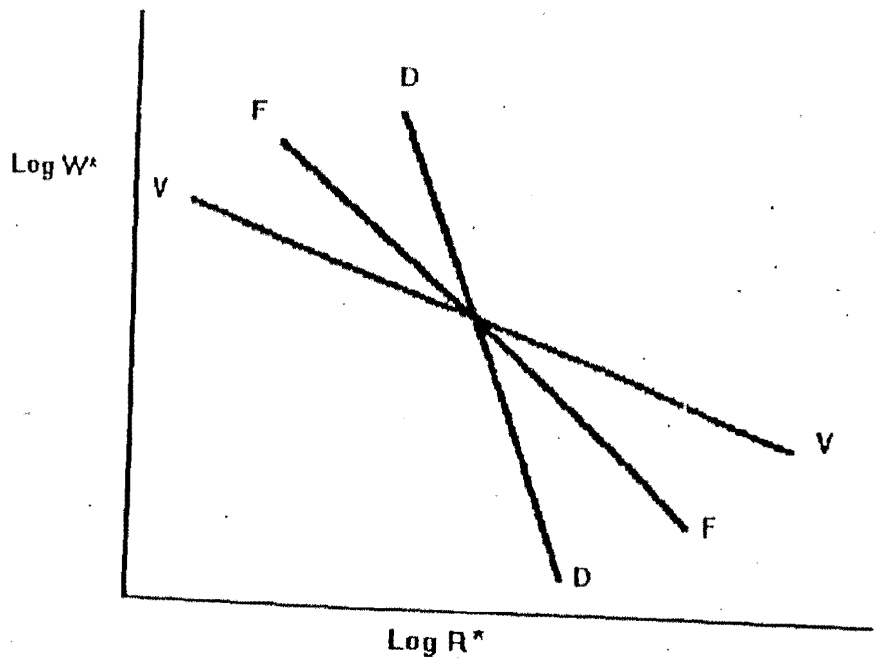

FIGURE 3.