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Existence and Optimality of Mediation Schemes for Games with Communication

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ABSTRACT

This paper explicitly considers the possibility of a mediator selecting equilibria in incomplete information environments. We study two models of mediated communication in incomplete information games; the first model with a disinterested mediator and second with an interested mediator. The notion of *ex post* incentive compatibility (EPIC) is defined as a solution concept for such models and its existence is shown in very general settings. The second main result shows the existence of an optimal EPIC when optimality is defined in terms of the mediator's welfare.

1. Statement of problem

It is folk wisdom that Nash-like equilibrium concepts (including refinements of Nash equilibrium, Bayesian equilibrium, etc.) are *consistency* conditions on the predicted strategy choices in a game, but do not in themselves contain a theory of *how* players might arrive at the predicted choices. These questions are traditionally dealt with by an appeal to supplementary devices such as "focal points" or "common knowledge". Such explanations, however, merely beg the question. In any game, especially in games with multiple equilibria, neither approach can satisfactorily explain the predicted *correlation* of players choices. One can identify at least three strands in the literature that attempts to deal with this problem.

The refinement approach imposes ever stronger rationality criteria in order to prune the set of equilibria. This literature, by and large, does not stray too far from the essence of non-cooperative game theory, though some of it skates quite close to the edge. Leaving aside the controversies regarding the appropriate definition of rationality and other foundational questions, it should be noted that the refinement program is not designed to address the *procedural* problems alluded to above that make the interpretation of *all* Nash-like equilibrium concepts (and their refinements) problematic. The spirit behind the refinement approach to defending predictions is akin to the Holmesian dictum asserting that after one has eliminated all other possibilities, whatever remains must be the truth. Drawing the obvious parallel with our problem, the fact that a prediction survives the most excruciatingly demanding refinement is irrelevant as our question is not "What is the truth?", but "Why is the truth the truth?"

A second strand is the evolutionary approach that radically alters the description and interpretation of a non-cooperative game by, say, postulating random interaction of large populations of genetically programmed players. Equilibrium in such models is no longer the result of cogitation by rational players, but the stable or ergodic outcome of a dynamical system that is driven, in a manner of speaking, by the animal spirits and genetic hardwiring of a large population of players. In this approach, the procedural issues mentioned above are either rendered trivial, as a player's 'genotype' completely determines his behaviour, or meaningless, as behaviour is subject to capricious "genetic mutations".

In common with the literature on evolutionary models, the third strand dispenses with a purely non-cooperative approach to the problem, in this case by breaching the traditionally hermetic boundary between cooperative and non-cooperative game theory.

This is done by supplementing the description of the game with a disinterested 'mediator' who communicates with the players. This approach takes the procedural problems head-on; while the decision problems faced by players in this approach might not demand the recondite eductive tâtonnements necessitated by Nash equilibrium and its refinements, it also avoids trivializing the decision problem.

In this paper, we supplement a given Bayesian game by adding a mediator to it; in our first model, the mediator is disinterested as his actions are not motivated by payoff considerations, but our second model concerns games with an interested mediator. The mediator is the hub of a communication system which allows the players to communicate with the mediator but not directly with each other. We shall consider a communication system in which players report their private information to the mediator and the mediator issues instructions to the players regarding their action choice.

When a mediator is added to a complete information normal form game, the natural solution concept is a correlated equilibrium (see Aumann, 1974, and Forges, 1986). In this case, the mediator instructs the players to perform certain actions and the players have to decide whether to obey the instructions. The problem for the mediator is to find instructions that the players have an incentive to obey. If the mediator's instructions are issued publicly, then each player has an incentive to obey, given the obedience of the other players, if and only if the profile of instructions is a Nash equilibrium of the given normal form game. By randomly choosing such profiles of instructions, the mediator can induce arbitrary probability distributions over the set of Nash equilibria. If the mediator issues instructions privately, then the instruction received by a player becomes private information, albeit payoff-irrelevant, for that player. As in a Bayesian equilibrium, although the private signal is privileged information, each player has to know the mapping used by the mediator to generate the private signals of all the players. Given the knowledge of this mapping, the players can decide whether to obey the instruction. By converting the given complete information game to an incomplete information game, the mediator can induce distributions over the space of payoff profiles that cannot be induced via public instructions.

The mediator's problem is more complicated in an incomplete information environment. In order to provide players appropriate incentives to obey the mediator, the mediator has to know the players' private information. While the mediator faces a pure moral hazard problem in a complete information game, the necessity of inducing the players to truthfully report their private information in an incomplete information game confronts the mediator with a combined moral hazard and adverse selection problem. Various stratagems have been used to study such games. One method of attack is to assume that a player's private information report is not verifiable but his action choices are alienable. Actions being alienable means that they can be contractually determined, or that the control over a player's action following a recommendation by the mediator is delegated to an agent, regulator, or machine. For instance, the alienation of a player's control over his actions is implicit in the definition of a direct mechanism as a mapping from the space of profiles of players' characteristics to the outcome space. If this alienation is by external fiat, then we have a *Bayesian collective choice problem*; if it is by personal choice of the player, then we have a *Bayesian bargaining problem* (see Myerson, 1991, for elucidation and references). In either case, the mixed problem is reduced to a pure adverse selection problem.

If a player's private information report is verifiable but his action choices are inalienable, then we have a *moral hazard problem modified by private information.* In Section 2, we define a solution concept for such problems when the mediator is disinterested, and in Section 3, we show the existence of such equilibria in very general circumstances. In Section 4 we show the existence of an optimal mediation plan when the mediator is interested. Some lemmata have been stated and proved in the Appendix.

2. Moral hazard modified by private information

There are many examples of moral hazard problems modified by private information. For instance, negotiations mediated by an "honest broker" often involve parties with private information which can be verified by the mediator. The outcomes in such situations often take the form of publicly announced "agreements" that specify the actions to be performed by the various negotiating parties. Such situations include: (a) international negotiations involving a third country or an international agency acting as a mediator, verifier and/or monitor, (b) labor/trade disputes mediated by the courts or some other mediating organization, (c) financial and commercial negotiations or disputes mediated by the courts or some other agency, (d) marriage counselling. In all these cases, the mediator typically has access to private information: e.g. the size and quality of a country's nuclear/chemical/biological arsenal, economic data for a country that may not be publicly available, financial records of a negotiating firm, the market projections of a firm, a spouse's peccadilloes. Another class of examples results from the problem faced by a principal who has to coordinate the efforts of many agents working on a project. Each agent could have private information that the principal can verify but which, for various reasons (such as competition for priority in publication, patents, promotions, etc.), is not shared by all agents. Such problems are routine for coordinators of scientific research projects that involve multi-tasking and managers of large diversified firms who have to control the activities of the managers of the firms various constituent units.

An important feature of such problems is that the mediator's recommended action for a given player can rely on information that was hitherto unavailable to that player. It can be argued that the desire to exploit these potential information spillovers is the principal reason for introducing a mediator in the examples cited above. The general hybrid case in which information is not verifiable and actions are not alienable raises very difficult conceptual problems. Attempts to study such problems have been made in very special models (see Forges, 1986, and Myerson, 1991), but we are unaware of any general method of attack.

We begin by setting up the standard incomplete information framework; some notational conventions are stated in Section 3. Let $(T, \mathcal{T}, \mu)$ be a probability space, where $\mathcal{T}$ is a σ-algebra on state space T, and μ is a probability measure on $(T, \mathcal{T})$. N is the set of players. $(T_i, \mathcal{T}_i)$ is a measurable space, where T_i is the space of signals sent by Nature to player $i \in N$ and $\mathcal{T}_i$ is a σ-algebra on T_i. $\theta_i : T \to T_i$ is the measurable surjective mapping that generates player i's signal. Let $\sigma(\theta_i)$ be the σ-algebra generated on T by θ_i. We refer to $\sigma(\theta_i)$ as player i's information. Let $\theta : T \to \prod_{i \in N} T_i$ be given by $\theta(t) = (\theta_i(t))_{i \in N}$. In the language of the canonical incomplete information model, T_i is player i's type space; given $T = \prod_{i \in N} T_i$, θ_i is the projection from T to T_i; μ is the common prior.

A_i is player i's action space, $A = \prod_{i \in N} A_i$ is the space of action profiles, and $A_{-i} = \prod_{j \in N - \{i\}} A_j$. Player i's utility function is $u_i : T \times A \to \Re$. Collecting the above data, we have a Bayesian game

$$\Gamma = \langle (T, \mathcal{T}, \mu), N, ((T_i, \mathcal{T}_i), \theta_i, A_i, u_i)_{i \in N} \rangle. \tag{2.1}$$

We assume that Γ is common knowledge. Let $M(X)$ denote the set of probability measures on a measurable space $(X, \mathcal{X})$. Given $(T, \mathcal{T}, \mu)$, $(T_i, \mathcal{T}_i)$ and θ_i, player i's conditional distribution on $(T, \mathcal{T})$ is given by $\Lambda_i : T_i \times \mathcal{T} \to [0,1]$, where $\Lambda_i(t_i) \in M(\mathcal{T})$ for every $t_i \in T_i$ and $\Lambda_i(E) : T_i \to [0,1]$ is $\mathcal{T}_i/\mathcal{B}([0,1])$ measurable for every $E \in \mathcal{T}$. Let A^T be the space of measurable functions from T to A. Given $a \in A^T$, let $a_{-i} : T \to A_{-i}$ be defined by $a_{-i}(t) = (a(t))_{-i}$. Suppose player i's information in state t is $I_i(t) = (\theta_i, \theta_i(t), a)$; i.e. he knows the mappings θ_i and a, and the realization $\theta_i(t)$. The expected utility function conditional on $I_i(t)$ is $U_i : T \times A^T \times A_i \to \Re$, given by the formula

$$U_i(t, a, b) = \int_T \Lambda_i(\theta_i(t), ds) u_i(s, a_{-i}(s), b),$$

for $(t, a, b) \in T \times A^T \times A_i$. Alternatively, suppose i's information in state t is $J_i(t) = (\theta_i, \theta_i(t), a(t))$; i.e. he knows the mapping θ_i and the realizations $\theta_i(t)$ and $a(t)$. The expected utility function conditional on $J_i(t)$ is $v_i : T \times A \to \Re$, given by the formula

$$v_i(t, a) = \int_T \Lambda_i(\theta_i(t), ds) u_i(s, a), \tag{2.2}$$

for $(t, a) \in T \times A$.

Definition 2.3. *A Bayesian Equilibrium (BE) for Γ is a mapping $a : T \to A$ such that, for every $i \in N$*

(a) the component mapping $a_i : T \to A_i$ is $\sigma(\theta_i)/\mathcal{B}(A_i)$ measurable, and

(b) $U_i(t, a, a_i(t)) \geq U_i(t, a, b)$ for every $(t, b) \in T \times A_i$.

Condition (a) ensures that player i's strategy a_i respects the private information structure defined above, i.e. knowing $a_i(t)$ in state t yields player i no more information about the true state than he can deduce from the signal $\theta_i(t)$. Condition (b) guarantees that in every state t that player i can distinguish using his private information, player i is implementing the best possible action.

Suppose we add to Γ a mediator. The mediator knows the functions $(\theta_i)_{i \in N}$ and recommends actions to the players using a *mediation plan* $a : T \to A$. His information about the state is derived from observing the profile of signals $(\theta_i(t))_{i \in N}$; we assume that the players report their signals to the mediator who can verify them with perfect accuracy. Thus, we are assuming away the adverse selection problem. The game now is played in four steps:

(a) Nature picks state $t \in T$ and sends private information $\theta_i(t)$ to player i.

(b) Player i reports $\theta_i(t)$ to the mediator. Seeing $(\theta_i(t))_{i \in N}$ and knowing $(\theta_i)_{i \in N}$, the mediator's information is the join of all the players' private informations, represented by the σ-algebra $\vee_{i \in N} \sigma(\theta_i)$.

(c) The mediator recommends action $a_i(t)$ to player i.

(d) Given $a_i(t)$, player i decides whether to obey or to deviate. A strategy for player i specifies the choice of an action in A_i for every signal-recommendation pair $(\theta_i(t), a_i(t))$. Given θ_i and a, this amounts to selecting a mapping $d_i : T \to A_i$.

Given the communication system described above, consider how a mediator can implement a BE a for the game Γ.[1] Before the game begins, the players are publicly informed that mediation plan a will be used; recall that a is required to be common knowledge in the standard interpretation of BE. The game is played as specified above. On getting the players' reports, the mediator privately recommends action $a_i(t)$ to player i. Thus, when selecting an action, player i knows the mappings (θ_i, a) and the signal-recommendation pair $(\theta_i(t), a_i(t))$. Knowledge of $(a, a_i(t))$ does not reveal to player i any information about the state beyond what can be deduced from $(\theta_i, \theta_i(t))$ because, by the definition of a BE, a_i is measurable with respect to player i's private information. Therefore, player i's distribution on $(T, \mathcal{T})$, conditional on his information, is $\Lambda_i(\theta_i(t))$. If all the other players are always obedient and player i chooses action $d_i(t)$ in state t, then his conditional expected utility is $U_i(t, a, d_i(t))$. As a is a BE, we have $U_i(t, a, a_i(t)) \geq U_i(t, a, b)$ for all $i \in N$, $t \in T$, and $b \in A_i$. Thus, every player will be obedient in every state, assuming other players are always obedient.

In the above-described implementation of a BE by a mediator, the mediator does not fully use the available information. More precisely, the mediator uses the profile $(\theta_i(t))_{i \in N}$ to correlate the implementation of a particular BE, but does not correlate the players actions in a way that allows private information 'spillovers', i.e. by recommending actions that are not measurable with respect to their private informations. The distribution induced by a BE a on the space of action profiles A is $\prod_{i \in N} \mu \circ a_i^{-1}$, where each a_i is required to be measurable with respect to player i's private information. As in the case of a correlated equilibrium, the mediator in an incomplete information setting may attempt to expand the set of possible distributions on A by choosing recommendations a_i that are not necessarily measurable with respect to player i's private information. We proceed to such a model.

Let $a : T \to A$ be a $\vee_{i \in N} \sigma(\theta_i)/\mathcal{B}(A)$ measurable mapping. Unlike in the case of a BE, the component mappings a_i are not required to be $\sigma(\theta_i)/\mathcal{B}(A_i)$ measurable. Suppose a is the publicly announced mediation plan. Let steps (a) and (b) of the game be exactly as described above. Consider player i's decision problem after getting the signal $\theta_i(t)$ but before hearing the mediator's recommendation. If all players are obedient and player i

[1] Although the standard definition of a Bayesian equilibrium does not involve a mediator, the notion of a mediator is a useful interpretational construct in order to explain how a particular equilibrium is selected out of a set of Bayesian equilibria and how this selection becomes common knowledge.

uses strategy d_i, then i's conditional expected utility is $\int_T \Lambda_i(\theta_i(t), ds)u_i(s, a_{-i}(s), d_i(s))$. Thus, in order to induce *ex ante* obedience from all players in all states t, a must satisfy the condition: for every $i \in N$ and $(t, d_i) \in T \times A_i^T$

$$\int_T \Lambda_i(\theta_i(t), ds)u_i(s, a(s)) \geq \int_T \Lambda_i(\theta_i(t), ds)u_i(s, a_{-i}(s), d_i(s)). \tag{2.4}$$

We call (2.4) an *ex ante* incentive condition because it induces obedience *before* the mediator's advice is received. However, the relevant question is whether a induces obedience *after* the mediator's recommendations are sent. (2.4) does not guarantee this. Once the mediator's advice is seen, each players will invert this message by using his knowledge of the mediation plan a. Consequently, player i's conditional distribution on $(T, \mathcal{T})$ after getting the mediator's advice will be $\tilde{\Lambda}_i(\theta_i(t), a_i(t))$. This problem does not arise for a BE, as in that case the mediator's advice is measurable with respect to the player's private information, i.e. $\Lambda_i(\theta_i(t)) = \tilde{\Lambda}_i(\theta_i(t), a_i(t))$.

Clearly, if player i knows (a) the mediation plan a, (b) the signal-recommendation pair $(\theta_i(t), a_i(t))$, and (c) a_i is not $\sigma(\theta_i)/\mathcal{B}(A_i)$ measurable, then i's private information $\sigma(\theta_i, a_i)$ is finer than $\sigma(\theta_i)$, which creates the sequential rationality problem. This implies a basic conflict between (a) and (c). Therefore, one route out of this dilemma is to drop (c) by making a_i $\sigma(\theta_i)/\mathcal{B}(A_i)$ measurable, in which case we are back to BE. The other resolution is to drop (a) by making the mediator *inscrutable*, i.e. the players do not know the mediation plan a. If the mediator wishes to go beyond BE, then the second route is the one to choose.

Let $a : T \to A$ be a $\vee_{i \in N}\sigma(\theta_i)/\mathcal{B}(A)$ measurable mapping and suppose the mediator is inscrutable. We alter the mediation process as follows. Let steps (a) and (b) of the game be exactly as described above. Step (c), however, is different: the mediator publicly announces $a(t)$ with player i advised to implement $a_i(t)$. Knowing $\theta_i(t)$ and $a(t)$, player i decides whether to obey the mediator. In the two earlier cases, as a was common knowledge, the players could use a to generate a belief about the other players actions. When the players do not know a, it is necessary to specify the players beliefs about each others actions in another way. The simplest alternative is to specify that the mediator's recommendations profile $a(t)$ is common knowledge. As player i does not know a, his conditional distribution on $(T, \mathcal{T})$ is $\Lambda_i(\theta_i(t))$. If the other players are assumed to be obedient, then i's expected utility from implementing action $b \in A_i$ in state t, conditional on seeing $(\theta_i(t), a(t))$ and knowing θ_i, is $v_i(t, a_{-i}(t), b)$. Thus, we have the following solution concept.

Definition 2.5. *An ex post incentive compatible equilibrium (EPIC) for Γ is a mapping $a : T \to A$ such that*

(a) a is $\vee_{i \in N} \sigma(\theta_i)/\mathcal{B}(A)$ measurable, and

(b) $v_i(t, a(t)) \geq v_i(t, a_{-i}(t), b)$, for every $i \in N$ and $(t, b) \in T \times A_i$.

3. Existence of an EPIC

We begin by stating some conventions. $F : X \Rightarrow Y$ will denote a mapping with domain X and values in 2^Y. $\Re$ is given the Euclidean metric topology. For a topological space X, the relevant σ-algebra is the Borel σ-algebra, denoted $\mathcal{B}(X)$. Subsets of topological (resp. measurable) spaces are given the subspace topology (resp. trace σ-algebra). Products of topological (resp. measurable) spaces are given the product topology (resp. σ-algebra). As long as the involved spaces are separable, there is no conflict among these conventions. If $\{\mathcal{X}_i \mid i \in I\}$ is a family of σ-algebras on a set X, then $\vee_{i \in I} \mathcal{X}_i$ denotes the σ-algebra generated by $\cup_{i \in I} \mathcal{X}_i$. If $(X, \mathcal{X})$ is a measurable space and Y is a topological space, then $F : X \Rightarrow Y$ is called measurable (resp. weakly measurable) with respect to $\mathcal{X}$ if $\{x \in X \mid F(x) \cap E \neq \emptyset\} \in \mathcal{X}$ for every E closed (resp. open) in Y. If X is a measurable space, then $M(X)$ denotes the space of probability measures on X; if X is topological, then $M(X)$ will be given the weak* topology. We start with the following lemma.

Lemma 3.1. *Given $u : T \times A \to \Re$, let $U : M(T) \times A \to \Re$ be defined by $U(\lambda, a) = \int_T \lambda(dt) u(t, a)$ for $(\lambda, a) \in M(T) \times A$.*

(a) If T and A are compact metric, and u continuous, then U is continuous.

(b) If $u(t, a_{-i}, .) : A_i \to \Re$ is concave for every $(t, a_{-i}) \in T \times A_{-i}$, then $U(\lambda, a_{-i}, .) : A_i \to \Re$ is concave for every $(\lambda, a_{-i}) \in M(T) \times A_{-i}$.

Proof. Part (a) follows from Lemma II.6.1 in Parthasarathy (1967) and Lemma A.1. Part (b) is routine.

The following result provides sufficient conditions for the existence of an EPIC.

Theorem 3.2. *Consider $\Gamma = \langle (T, \mathcal{B}(T), \mu), N, ((T_i, \mathcal{B}(T_i)), \theta_i, A_i, u_i)_{i \in N} \rangle$. Suppose*

(a) T is compact metric,

(b) N is countable, and

for every $i \in N$,

(c) $(T_i, \mathcal{B}(T_i))$ is a separable standard Borel space,

(d) A_i is a nonempty, convex, compact and metrizable subset of a locally convex linear

topological space, and

(e) $U_i : M(T) \times A \to \Re$, defined by $U_i(\lambda, a) = \int_T \lambda(dt) u_i(t, a)$ for $(\lambda, a) \in M(T) \times A$, is continuous and $U_i(\lambda, a_{-i}) : A_i \to \Re$ is quasi-concave for all $(\lambda, a_{-i}) \in M(T) \times A_{-i}$. Then there exists an EPIC for Γ.

Proof. Define $V_i : M(T) \times A \to \Re$ by $V_i(\lambda, a) = \max\{U_i(\lambda, a_{-i}, b) \mid b \in A_i\}$ for $(\lambda, a) \in M(T) \times A$. V_i is continuous (Berge, 1963, Theorems VI.3.1 and VI.3.2). Define $B_i : M(T) \times A \Rightarrow A_i$ by $B_i(\lambda, a) = \{b \in A_i \mid U_i(\lambda, a_{-i}, b) - V_i(\lambda, a) \geq 0\}$ for $(\lambda, a) \in M(T) \times A$. As U_i and V_i are continuous, $\mathrm{Gr}\, B_i = \{(\lambda, a, b) \in M(T) \times A \times A_i \mid U_i(\lambda, a_{-i}, b) - V_i(\lambda, a) \geq 0\}$ is closed. Consequently, B_i is u.s.c. (Berge, 1963, Theorem VI.1.7). Moreover, B_i has nonempty, convex and compact values. Define $B : M(T)^N \times A \Rightarrow A$ by $B(\lambda, a) = \prod_{i \in N} B_i(\lambda_i, a)$ for $(\lambda, a) \in M(T)^N \times A$. B is u.s.c. (Fan, 1952, Lemma A.4), with nonempty, convex and compact values. Define $\Xi : M(T)^N \Rightarrow A$ by $\Xi(\lambda) = \{a \in A \mid a \in B(\lambda, a)\}$ for $\lambda \in M(T)^N$. Ξ has nonempty values (Browder, 1968, Theorem 4). By Lemma A.7, there is a $\mathcal{B}(M(T)^N)/\mathcal{B}(A)$ measurable function $\xi : M(T)^N \to A$ such that $\xi(\lambda) \in \Xi(\lambda)$ for every $\lambda \in M(T)^N$.

As T is compact metric, $(T, \mathcal{B}(T))$ is a separable standard Borel space. Thus, by Lemma A.2, for every $i \in N$, there exists $\Lambda_i : T_i \times \mathcal{B}(T) \to [0, 1]$ that satisfies the properties listed in Lemma A.2(A). By Lemma A.2(B), $\hat{\Lambda}_i : T_i \to M(T)$, defined by $\hat{\Lambda}_i(t_i)(.) = \Lambda_i(t_i, .)$, is $\mathcal{B}(T_i)/\mathcal{B}(M(T))$ measurable. As θ_i is $\sigma(\theta_i)/\mathcal{B}(T_i)$ measurable, $\hat{\Lambda}_i \circ \theta_i$ is $\sigma(\theta_i)/\mathcal{B}(M(T))$ measurable. Therefore, $\hat{\Lambda}_i \circ \theta_i$ is $\vee_{i \in N} \sigma(\theta_i)/\mathcal{B}(M(T))$ measurable for every $i \in N$. Define $\Lambda : T \to M(T)^N$ by $\Lambda(t) = (\hat{\Lambda}_i \circ \theta_i(t))_{i \in N}$. Λ is $\vee_{i \in N} \sigma(\theta_i)/\mathcal{B}(M(T))^N$ measurable, where $\mathcal{B}(M(T))^N$ is the product σ-algebra on $M(T)^N$. As $M(T)$ is compact metric, it is separable. Therefore, $\mathcal{B}(M(T))^N = \mathcal{B}(M(T)^N)$ (Parthasarathy, 1967, Theorem I.1.10). Thus, $a = \xi \circ \Lambda$ is $\vee_{i \in N} \sigma(\theta_i)/\mathcal{B}(A)$ measurable. It is easy to confirm that a is an EPIC.

Remark. Lemma 3.1 provides sufficient conditions on Γ for condition (e) of Theorem 3.2 to hold.

Theorem 3.2 requires compactness of T and continuity of u_i. In Theorem 3.4, (a) T is not required to be compact, and (b) utility functions are Caratheodory rather than continuous. However, some weaker topological structure on T is retained in order to guarantee the existence of regular conditional distributions for the players.

Lemma 3.3. *Consider $\Gamma = \langle (T, \mathcal{T}, \mu), N, ((T_i, \mathcal{T}_i), \theta_i, A_i, u_i)_{i \in N} \rangle$. Suppose*

(a) $(T, \mathcal{T}, \mu)$ is a probability space,

(b) *N is countable, and*

for every $i \in N$,

(c) $(T_i, \mathcal{T}_i)$ *is a measurable space,*

(d) $\theta_i : T \to T_i$ *is a measurable surjection and* $(T, \vee_{i \in N}\sigma(\theta_i))$ *is complete with respect to* μ,

(e) A_i *is a nonempty, convex and compact subset of a Banach space, and*

(f) $v_i : T \times A \to \Re$, *defined by (2.2), is such that (i)* $v_i(t) : A \to \Re$ *is continuous for every* $t \in T$, *(ii)* $v_i(a) : T \to \Re$ *is* $\sigma(\theta_i)/\mathcal{B}(\Re)$ *measurable for every* $a \in A$, *and (iii)* $v_i(t, a_{-i}) : A_i \to \Re$ *is quasi-concave.*

Then there exists a mapping $\phi : T \times A \Rightarrow A$ *such that*

(A) ϕ *has nonempty, compact and convex values,*

(B) ϕ *is measurable with respect to* $\vee_{i \in N}\sigma(\theta_i) \otimes \mathcal{B}(A)$,

(C) $\phi(t) : A \Rightarrow A$ *is u.s.c. for every* $t \in T$, *and*

(D) $a : T \to A$ *is an EPIC iff.* a *is* $\vee_{i \in N}\sigma(\theta_i)/\mathcal{B}(A)$ *measurable and* $a(t) \in \phi(t, a(t))$ *for every* $t \in T$.

Proof. Until step (5), fix $i \in N$.

(1) Define $V_i : T \times A \times A_i \to \Re$ by $V_i(t, a, b) = v_i(t, a_{-i}, b) - v_i(t, a)$ for $(t, a, b) \in T \times A \times A_i$. It follows from (f) that: (i) $V_i(t) : A \times A_i \to \Re$ is continuous for every $t \in T$, (ii) $V_i(t, b) : A \to \Re$ is continuous for every $(t, b) \in T \times A_i$, (iii) $V_i(t, a) : A_i \to \Re$ is continuous for every $(t, a) \in T \times A$, and (iv) $V_i(a, b) : T \to \Re$ is $\sigma(\theta_i)/\mathcal{B}(\Re)$ measurable for every $(a, b) \in A \times A_i$. (i) and (iv) imply that V_i is $\sigma(\theta_i) \otimes \mathcal{B}(A \times A_i)/\mathcal{B}(\Re)$ measurable (Himmelberg, 1975, Theorem 6.1). Analogously, (ii) and (iv) imply that $V_i(b)$ is $\sigma(\theta_i) \otimes \mathcal{B}(A)/\mathcal{B}(\Re)$ measurable for every $b \in A_i$.

(2) Define $F_i : T \times A \Rightarrow A_i$ by $F_i(t, a) = \{b \in A_i \mid V_i(t, a, b) > 0\}$. It follows from (f) that F_i has convex values. Step (1) and Lemma A.3 imply that $F_i(t) : A \Rightarrow A_i$ is lower semicontinuous for every $t \in T$. We show that F_i is weakly measurable with respect to $\sigma(\theta_i) \otimes \mathcal{B}(A)$. As A_i is compact metric, it is separable. Let C be a countable set that is dense in A_i. Let E be open in A_i. Then,

$$\begin{aligned}\{(t, a) \in T \times A \mid F_i(t, a) \cap E \neq \emptyset\} &= \{(t, a) \in T \times A \mid \exists b \in E : V_i(t, a, b) > 0\} \\ &= \{(t, a) \in T \times A \mid \exists b \in E \cap C : V_i(t, a, b) > 0\} \\ &= \cup_{b \in E \cap C}\{(t, a) \in T \times A \mid V_i(t, a, b) > 0\}.\end{aligned}$$

By (1), $\{(t, a) \in T \times A \mid V_i(t, a, b) > 0\} \in \sigma(\theta_i) \otimes \mathcal{B}(A)$ for every $b \in E \cap C$. Thus, $\{(t, a) \in T \times A \mid F_i(t, a) \cap E \neq \emptyset\} \in \sigma(\theta_i) \otimes \mathcal{B}(A)$.

(3) Let $D_i = \{(t,a) \in T \times A \mid F_i(t,a) \neq \emptyset\}$. As F_i is weakly measurable with respect to $\sigma(\theta_i) \otimes \mathcal{B}(A)$, $D_i = \{(t,a) \in T \times A \mid F_i(t,a) \cap A_i \neq \emptyset\} \in \sigma(\theta_i) \otimes \mathcal{B}(A)$. Define $D_i^t = \{a \in A \mid (t,a) \in D_i\}$ for $t \in T$. Clearly, $D_i^t = \cup_{b \in A_i}\{a \in A \mid V_i(t,a,b) > 0\}$. Using step (1), $\{a \in A \mid V_i(t,a,b) > 0\}$ is open in A for every $(t,b) \in T \times A_i$. Thus, D_i^t is open in A for every $t \in T$.

Let $D_i^a = \{t \in T \mid (t,a) \in D_i\}$ for $a \in A$. Clearly, $D_i^a = \{t \in T \mid F_i(t,a) \neq \emptyset\} = \{t \in T \mid \exists b \in A_i : V_i(t,a,b) > 0\}$. As $V_i(t,a)$ is continuous by step (1), $D_i^a = \{t \in T \mid \exists b \in C : V_i(t,a,b) > 0\} = \cup_{b \in C}\{t \in T \mid V_i(t,a,b) > 0\}$. As $V_i(a,b)$ is $\sigma(\theta_i)/\mathcal{B}(\Re)$ measurable by step (1), $\{t \in T \mid V_i(t,a,b) > 0\} \in \sigma(\theta_i)$ for every $(a,b) \in A \times A_i$. Therefore, $D_i^a \in \sigma(\theta_i)$ for every $a \in A$. Given $(t,a) \in T \times A$, $V_i(t,a) : A_i \to \Re$ is continuous by step (1). Therefore, for $(t,a) \in D_i$, $F_i(t,a)$ is nonempty and open in A_i. It follows from Theorem 3.2 in Kim, Prikry and Yannelis (1987) that there exists $f_i : D_i \to A_i$ such that (i) $f_i(t,a) \in F_i(t,a)$ for every $(t,a) \in D_i$, (ii) $f_i(t) : D_i^t \to A_i$ is continuous for every $t \in T$, and (iii) $f_i(a) : D_i^a \to A_i$ is $(\sigma(\theta_i) \cap D_i^a)/\mathcal{B}(A_i)$ measurable for every $a \in A$.

(4) Define the map $\phi_i : T \times A \Rightarrow A_i$ by the formula

$$\phi_i(t,a) = \begin{cases} \{f_i(t,a)\}, & \text{if } (t,a) \in D_i \\ A_i, & \text{if } (t,a) \in (T \times A) - D_i. \end{cases}$$

By Lemma A.4, $\phi_i(t) : A \Rightarrow A_i$ is u.s.c. for every $t \in T$. By Lemma A.5, ϕ_i is weakly measurable with respect to $\sigma(\theta_i) \otimes \mathcal{B}(A)$, and therefore with respect to $\vee_{i \in N}\sigma(\theta_i) \otimes \mathcal{B}(A)$. Moreover, ϕ_i has nonempty, convex and compact values.

(5) Define $\phi : T \times A \Rightarrow A$ by the formula $\phi(t,a) = \prod_{i \in N} \phi_i(t,a)$. We now confirm our claims. (A) ϕ has nonempty and convex values. As ϕ_i has compact values for every $i \in N$, ϕ has compact values. (B) By step (4) and Lemma A.8, ϕ is weakly measurable with respect to $\vee_{i \in N}\sigma(\theta_i) \otimes \mathcal{B}(A)$; consequently, ϕ is measurable with respect to $\vee_{i \in N}\sigma(\theta_i) \otimes \mathcal{B}(A)$ (Himmelberg, 1975, Theorem 3.5(ii)). (C) Given that $\phi_i(t)$ is u.s.c. for every $i \in N$, and N is countable, $\phi(t) : A \Rightarrow A$ is u.s.c. for every $t \in T$ (Fan, 1952, Lemma A.4). (D) Suppose $a : T \to A$ is a $\vee_{i \in N}\sigma(\theta_i)/\mathcal{B}(A)$ measurable function such that $a(t) \in \phi(t, a(t))$ for every $t \in T$. Fix $t \in T$ and $i \in N$. By construction, $a_i(t) \in \phi_i(t, a(t))$. If $(t, a(t)) \in D_i$, then $a_i(t) = f_i(t, a(t)) \in F_i(t, a(t))$. This implies $V_i(t, a(t), a_i(t)) > 0$, which is a contradiction. So, $(t, a(t)) \in (T \times A) - D_i$. This implies $F_i(t, a(t)) = \emptyset$. As this holds for every $t \in T$ and $i \in N$, a is an EPIC. Conversely, suppose $a : T \to A$ is an EPIC. By definition, a is $\vee_{i \in N}\sigma(\theta_i)/\mathcal{B}(A)$ measurable. Consider $t \in T$ and $i \in N$. If $(t, a(t)) \in D_i$, then $F_i(t, a(t)) \neq \emptyset$. Thus, there exists $b \in A_i$ such that $v_i(t, a_{-i}(t), b) > v_i(t, a(t))$, which

contradicts the fact that a is an EPIC. So, $(t, a(t)) \in (T \times A) - D_i$ for every $t \in T$ and $i \in N$. By the definition of ϕ_i, this implies $a_i(t) \in A_i = \phi_i(t, a(t))$. Thus, $a(t) \in \phi(t, a(t))$ for every $t \in T$.

We immediately have the following existence result.

Theorem 3.4. *Suppose $\Gamma = \langle (T, \mathcal{B}(T), \mu), N, ((T_i, \mathcal{B}(T_i)), \theta_i, A_i, u_i)_{i \in N} \rangle$ satisfies assumptions (a) to (e) of Lemma 3.3. In addition, suppose*

(f) $(T, \mathcal{B}(T))$ is a separable standard Borel space, and

for every $i \in N$,

(g) $(T_i, \mathcal{B}(T_i))$ is a separable standard Borel space, and

(h) $u_i : T \times A \to \Re$ is such that (i) $u_i(t) : A \to \Re$ is continuous for every $t \in T$, (ii) $u_i(a) : T \to \Re$ is $\mathcal{B}(T)/\mathcal{B}(\Re)$ measurable for every $a \in A$, and (iii) $u_i(t, a_{-i}) : A_i \to \Re$ is concave.

Then there exists an EPIC for Γ.

Proof. If Γ satisfies assumptions (a) to (f) of Lemma 3.3, then there exists $\phi : T \times A \rightrightarrows A$ with properties (A) to (D) listed in Lemma 3.3. Lemma A.6 implies the existence of a $\vee_{i \in N} \sigma(\theta_i)/\mathcal{B}(A)$ measurable function $a : T \to A$ such that $a(t) \in \phi(t, a(t))$ for every $t \in T$, which implies the existence of an EPIC by Lemma 3.3(D).

We check that assumption (f) of Lemma 3.3 is satisfied. For every $t \in T$, continuity of $u_i(t)$ implies that of $v_i(t)$. Since $u_i(t, a_{-i}) : A_i \to \Re$ is concave, $v_i(t, a_{-i}) : A_i \to \Re$ is concave. Fix $a \in A$. There exists a regular conditional distribution on $(T, \mathcal{B}(T))$ given θ_i, denoted by $\Lambda_i : T_i \times \mathcal{B}(T) \to [0, 1]$ (Parthasarathy, 1967, Theorem V.8.1). By the non-Cartesian version of the Fubini-Stone theorem (Rao, 1987, Exercise 6.2.3), the mapping $t_i \mapsto \int_T \Lambda_i(t_i, dt) u_i(t, a)$ is $\mathcal{B}(T_i)/\mathcal{B}(\Re)$ measurable. Since θ_i is $\sigma(\theta_i)/\mathcal{B}(T_i)$ measurable, composing these two mappings implies that $v_i(a)$ is $\sigma(\theta_i)/\mathcal{B}(\Re)$ measurable.

4. Existence of an optimal EPIC

Suppose the mediator has a welfare function $w : T \times A \to \Re$. In this case, given a game Γ, the mediator may wish to implement an optimal EPIC, i.e. an EPIC a such that, for every EPIC a' and state t, $w(t, a(t)) \geq w(t, a'(t))$. We show that this is possible very generally in the following result.

Theorem 4.1. *Consider $\Gamma = \langle (T, \mathcal{T}, \mu), N, ((T_i, \mathcal{T}_i), \theta_i, A_i, u_i)_{i \in N} \rangle$. If Γ satisfies assumptions (a) to (f) of Lemma 3.3, and $w : T \times A \to \Re$ is such that $w(t) : A \to \Re$ is continuous* *for every $t \in T$ and $w(a) : T \to \Re$ is $\vee_{i \in N} \sigma(\theta_i)/\mathcal{B}(\Re)$ measurable for every $a \in A$, then there exists an EPIC a such that, for every EPIC a' and state t, $w(t, a(t)) \geq w(t, a'(t))$.*

Proof. By Lemma 3.3, there exists a mapping $\phi : T \times A \Rightarrow A$ with properties (A) to (D) listed in Lemma 3.3. Define $\Phi : T \Rightarrow A$ by $\Phi(t) = \{a \in A \mid a \in \phi(t, a)\}$ and $\hat{\Phi} : T \Rightarrow \Re$ by $\hat{\Phi}(t) = w(\{t\} \times \Phi(t))$, for $t \in T$. Define $\hat{\phi} : T \to \Re$ by $\hat{\phi}(t) = \sup \hat{\Phi}(t)$. Suppose there exists an EPIC a such that $\hat{\phi}(t) = w(t, a(t))$ for every $t \in T$. Let a' be an EPIC. Fix $t \in T$. By Lemma 3.3(D), $a'(t) \in \Phi(t)$. Consequently, $w(t, a'(t)) \in \hat{\Phi}(t)$, and therefore, $w(t, a'(t)) \leq \hat{\phi}(t) = w(t, a(t))$. Thus, it is sufficient to show the existence of an EPIC a such that $\hat{\phi}(t) = w(t, a(t))$ for every $t \in T$.

By Lemma A.6, Φ is measurable, with nonempty compact values. Therefore, $\hat{\Phi}$ is weakly measurable with respect to $\vee_{i \in N} \sigma(\theta_i)$ (Himmelberg, 1975, Theorems 6.5), with nonempty compact values; indeed, $\hat{\Phi}$ is measurable as $\Re$ is σ-compact (Himmelberg, 1975, Theorem 3.5(ii)). It follows that $\hat{\phi}$ is measurable (Himmelberg, 1975, Theorem 6.6); moreover, as $\hat{\Phi}$ has compact values, $\hat{\phi}(t) \in w(\{t\} \times \Phi(t))$ for every $t \in T$. It follows that there exists a $\vee_{i \in N} \sigma(\theta_i)/\mathcal{B}(A)$ measurable function $a : T \to A$ such that $a(t) \in \Phi(t)$ and $\hat{\phi}(t) = w(t, a(t))$ for every $t \in T$ (Himmelberg, 1975, Theorem 7.1).

Note that this result does not merely maximize expected welfare but does so state-by-state. Consequently, optimality does not depend on the mediator's belief about the true state.

Appendix

Lemma A.1. *Suppose*

(a) X and Y are separable metric spaces, and

(b) $u : X \times Y \to \Re$ is bounded and continuous.

Then, $U : M(X) \times M(Y) \to \Re$, defined by $U(\mu, \lambda) = \int_{X \times Y} \mu \times \lambda(dx, dy) u(x, y)$, is continuous.

Proof. Let $V : M(X \times Y) \to \Re$ be defined by $V(\nu) = \int_{X \times Y} \nu(dz) u(z)$. As u is bounded and continuous, it is ν-integrable for every $\nu \in M(X \times Y)$. Let $f : M(X) \times M(Y) \to M(X \times Y)$ be defined by $f(\mu, \lambda) = \mu \times \lambda$. As $U = V \circ f$, it is sufficient to show the continuity of V and f. Continuity of V follows from the definition of a weak* topology. As X and Y are separable metric, so is $X \times Y$. Therefore, $M(X)$, $M(Y)$ and $M(X \times Y)$ are metrizable and separable (Parthasarathy, 1967, Theorem II.6.2). Consider sequences $(\mu_n) \subset M(X)$ and $(\lambda_n) \subset M(Y)$ converging to $\mu \in M(X)$ and $\lambda \in M(Y)$ respectively. Continuity of f follows if $\lim_{n \uparrow \infty} \mu_n \times \lambda_n = \mu \times \lambda$. This follows from Lemma III.1.1 in Parthasarathy (1967).

Lemma A.2. *Suppose*

(a) X and Y are metric spaces,

(b) $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ are separable standard Borel spaces, with Q a probability measure on $(X, \mathcal{B}(X))$, and

(c) $\pi : X \to Y$ is a measurable surjection.

(A) Then there exists a function $P : Y \times \mathcal{B}(X) \to [0,1]$ and $N \in \mathcal{B}(Y)$ such that (i) $Q \circ \pi^{-1}(N) = 0$, (ii) $P(y, \pi^{-1}(\{y\})) = 1$ for every $y \in Y - N$, and (iii) $Q(E) = \int_Y Q \circ \pi^{-1}(dy) P(y, E)$ for every $E \in \mathcal{B}(X)$.

(B) If, in addition to (a)-(c), X is compact, then $\hat{P} : Y \to M(X)$, defined by $\hat{P}(y)(.) = P(y, .)$, is $\mathcal{B}(Y)/\mathcal{B}(M(X))$ measurable.

Proof. (A) This follows from Theorem V.8.1 in Parthasarathy (1967).

(B) The space $[-1,1]^X$ of continuous functions $g : X \to [-1,1]$ with the compact-open topology is separable (Kuratowski, 1966, Theorem II.22.III). Let $\{f_i \mid i \in I\}$ be a countable dense subset of $[-1,1]^X$. Define $F : M(X) \to [-1,1]^I$ by $F(\mu) = \left(\int_X \mu(dx) f_i(x)\right)_{i \in I}$. As $\{f_i \mid i \in I\}$ is dense in $[-1,1]^X$, F is injective. By the definition of the weak* topology, F is continuous. As $M(X)$ is compact (Parthasarathy, 1967, Theorem II.6.4), F imbeds $M(X)$ in $[-1,1]^I$. As the Borel σ-algebras on $M(X)$ and $[-1,1]^I$ are generated by their respective topologies, it follows that $E \subset M(X)$ is measurable iff. $F(E) \subset [-1, 1]^I$ is measurable. Thus, $\hat{P}$ is measurable iff. $F \circ \hat{P}$ is measurable.

Also, $F \circ \hat{P}$ is measurable iff. $F_i \circ \hat{P} : Y \to [-1, 1]$ is measurable for every $i \in I$. By definition,

$$F_i \circ \hat{P}(y) = \int_X \hat{P}(y)(dx) f_i(x) = \int_X P(y, dx) f_i(x).$$

Measurability of $F_i \circ \hat{P}$ follows from the non-Cartesian version of the Fubini-Stone theorem (Rao, 1987, Exercise 6.2.3).

Lemma A.3. *Suppose X and Y are topological spaces, $X \times Y$ is given the product topology, and $g : X \times Y \to \Re$ is continuous. Then $G : X \Rightarrow Y$, defined by $G(x) = \{y \in Y \mid g(x, y) > 0\}$, has an open graph, and consequently, G is lower semicontinuous.*

Proof. $\mathrm{Gr}\, G = \{(x, y) \in X \times Y \mid y \in G(x)\} = \{(x, y) \in X \times Y \mid g(x, y) > 0\}$, which is open in $X \times Y$ as g is continuous.

To establish lower semicontinuity of G, fix $x \in X$. Let $V \subset Y$ be open in Y and $V \cap G(x) \neq \emptyset$. Let $y \in V \cap G(x)$. Therefore, $(x, y) \in (X \times V) \cap \mathrm{Gr}\, G$. As $\mathrm{Gr}\, G$ is open in $X \times Y$, $(X \times V) \cap \mathrm{Gr}\, G$ is open in $X \times Y$. Therefore, we can find E open in X and F open in Y such that $(x, y) \in E \times F$ and $E \times F \subset (X \times V) \cap \mathrm{Gr}\, G$. If $z \in E$, then $(z, y) \in (X \times V) \cap \mathrm{Gr}\, G$, i.e. $y \in V \cap G(z)$. Thus, $V \cap G(z) \neq \emptyset$ for every $z \in E$.

Lemma A.4. *Suppose*

(a) X and Y are topological spaces,

(b) $D \subset X$ is open in X, and

(c) $g : D \to Y$ is continuous with respect to the subspace topology.

If $\Gamma : X \Rightarrow Y$ is defined by

$$\Gamma(x) = \begin{cases} \{g(x)\}, & \text{if } x \in D, \\ Y, & \text{if } x \in X - D, \end{cases}$$

then Γ is u.s.c.

Proof. Suppose $x \in D$. Then $\Gamma(x) = \{g(x)\}$. Let E be an open neighborhood of $g(x)$. Given (c), there exists U open in X such that $x \in U$, and $y \in U \cap D$ implies $g(y) \in E$. As D is open in X, $U \cap D$ is open in X, and $y \in U \cap D$ implies $\Gamma(y) = \{g(y)\} \subset E$. Thus, Γ is u.s.c. at x. Suppose $x \in X - D$. Then $\Gamma(x) = Y$. As $y \in X$ implies $\Gamma(y) \subset Y$, Γ is u.s.c. at x.

Lemma A.5. *Suppose*

(a) $(\Omega, \mathcal{F})$ is a measurable space,

(b) X and Y are metric spaces with X separable,

(c) $D \in \mathcal{F} \otimes \mathcal{B}(X)$,

(d) $D_x = \{\omega \in \Omega \mid (\omega, x) \in D\} \in \mathcal{F}$ for every $x \in X$,

(e) $D_\omega = \{x \in X \mid (\omega, x) \in D\}$ is open in X for every $\omega \in \Omega$, and

(f) $g : D \to Y$ is such that $g(x) : D_x \to Y$ is $(\mathcal{F} \cap D_x)/\mathcal{B}(Y)$ measurable for every $x \in X$ and $g(\omega) : D_\omega \to Y$ is continuous for every $\omega \in \Omega$.

If $\Gamma : \Omega \times X \Rightarrow Y$ is defined by

$$\Gamma(\omega, x) = \begin{cases} \{g(\omega, x)\}, & \text{if } (\omega, x) \in D, \\ Y, & \text{if } (\omega, x) \in (\Omega \times X) - D, \end{cases}$$

then Γ is weakly measurable with respect to $\mathcal{F} \otimes \mathcal{B}(X)$.

Proof. Using the hypotheses, g is $[(\mathcal{F} \otimes \mathcal{B}(X)) \cap D]/\mathcal{B}(Y)$ measurable (Kim, Prikry and Yannelis, 1987, Lemma 4.12). Let E be an open subset of Y. Then

$$\{(\omega, x) \in \Omega \times X \mid \Gamma(\omega, x) \cap E \neq \emptyset\} = [(\Omega \times X) - D] \cup \{(\omega, x) \in D \mid g(\omega, x) \in E\}.$$

Given the measurability property of g, $\{(\omega, x) \in D \mid g(\omega, x) \in E\} = C \cap D$, where $C \in \mathcal{F} \otimes \mathcal{B}(X)$. As $D \in \mathcal{F} \otimes \mathcal{B}(X)$, we have $C \cap D \in \mathcal{F} \otimes \mathcal{B}(X)$. As $[(\Omega \times X) - D] \in \mathcal{F} \otimes \mathcal{B}(X)$, we have $\{(\omega, x) \in \Omega \times X \mid \Gamma(t, x) \cap E \neq \emptyset\} \in \mathcal{F} \otimes \mathcal{B}(X)$, which proves that Γ is weakly measurable with respect to $\mathcal{F} \otimes \mathcal{B}(X)$.

In the following two results, given $\Gamma : \Omega \times X \Rightarrow X$, the mapping $\Phi : \Omega \Rightarrow X$ is defined by $\Phi(\omega) = \{x \in X \mid x \in \Gamma(\omega, x)\}$ for $\omega \in \Omega$. If Ω and X are measurable, then $\phi \sim \Phi$ denotes that $\phi : \Omega \to X$ is a measurable function with $\phi(\omega) \in \Phi(\omega)$ for every $\omega \in \Omega$.

Lemma A.6. *Suppose*

(a) $(\Omega, \mathcal{F})$ is a complete measurable space,

(b) X is a nonempty, convex, compact and metrizable subset of a locally convex linear topological space,

(c) $\Gamma : \Omega \times X \Rightarrow X$ is weakly measurable with respect to $\mathcal{F} \otimes \mathcal{B}(X)$, and

(d) for every $\omega \in \Omega$, $\Gamma(\omega) : X \Rightarrow X$ is u.s.c., with nonempty, convex and closed values.

Then,

(A) Φ has nonempty closed values, and

(B) Φ is measurable and there exists $\phi \sim \Phi$.

Proof. (A) Fix $\omega \in \Omega$. Using (b) and (d), and applying Theorem 4 in Browder (1968) to $\Gamma(\omega)$, it follows that $\Phi(\omega) \neq \emptyset$.

We show that $\Phi(\omega)$ is closed in X. Suppose $x \in X - \Phi(\omega)$. We have to find an open neighborhood V of x such that $V \subset X - \Phi(\omega)$. As $x \in X - \Phi(\omega)$, we have $x \in X - \Gamma(\omega, x)$. By assumption, $\Gamma(\omega, x)$ is closed in X. By (b), X is regular; thus, there exist open neighborhoods U_1 of x and U_2 of $\Gamma(\omega, x)$, such that $U_1 \cap U_2 = \emptyset$. As $\Gamma(\omega)$ is u.s.c., there exists an open neighborhood U_3 of x such that $y \in U_3$ implies $\Gamma(\omega, y) \subset U_2$. Set $V = U_1 \cap U_3$. If $y \in V$, then $\Gamma(\omega, y) \subset U_2$. Since $U_2 \cap V = \emptyset$, this means $y \in X - \Gamma(\omega, y)$, i.e. $y \in X - \Phi(\omega)$. Thus, $V \subset X - \Phi(\omega)$.

(B) As Γ and the projection $(\omega, x) \mapsto x$ are weakly measurable with respect to $\mathcal{F} \otimes \mathcal{B}(X)$, the mapping $(\omega, x) \mapsto \Gamma(\omega, x) \cap \{x\}$ is weakly measurable with respect to $\mathcal{F} \otimes \mathcal{B}(X)$ (Himmelberg, 1975, Theorem 4.1). Thus, $\operatorname{Gr} \Phi = \{(\omega, x) \in \Omega \times X \mid x \in \Gamma(\omega, x)\} = \{(\omega, x) \in \Omega \times X \mid \Gamma(\omega, x) \cap \{x\} \cap X \neq \emptyset\} \in \mathcal{F} \otimes \mathcal{B}(X)$. Therefore, Φ is measurable (Himmelberg, 1975, Theorem 3.5(iii)). It follows that there exists $\phi \sim \Phi$ (Himmelberg, 1975, Theorem 5.1).

Lemma A.7. *Suppose*

(a) Ω is compact Hausdorff and X is compact metric,

(b) $\Gamma : \Omega \times X \Rightarrow X$ has a closed graph, and

(c) Φ has nonempty values.

Then, Φ is measurable and there exists $\phi \sim \Phi$.

Proof. Define $f : \Omega \times X \to \Omega \times \operatorname{diag} X^2$ by $f(\omega, x) = (\omega, x, x)$. As f is continuous and $\operatorname{Gr} \Phi = f^{-1}(\operatorname{Gr} \Gamma \cap (\Omega \times \operatorname{diag} X^2))$, it follows that $\operatorname{Gr} \Phi$ is closed. As $\Omega \times X$ is compact, $\operatorname{Gr} \Phi$ is compact. Consequently, Φ has compact values.

Let E be closed in X. Given the projection $\pi : (\omega, x) \mapsto \omega$, $\{\omega \in \Omega \mid \Phi(\omega) \cap E \neq \emptyset\} = \pi(\operatorname{Gr} \Phi \cap (\Omega \times E))$. As $\operatorname{Gr} \Phi \cap (\Omega \times E)$ is compact and π is continuous, $\{\omega \in \Omega \mid \Phi(\omega) \cap E \neq \emptyset\}$ is compact. As Ω is Hausdorff, $\{\omega \in \Omega \mid \Phi(\omega) \cap E \neq \emptyset\}$ is closed, and therefore, measurable. Since this holds for every E closed in X, Φ is measurable, and therefore, weakly measurable with respect to $\mathcal{B}(\Omega)$ (Himmelberg, 1975, Theorem 3.5(i)). The existence of $\phi \sim \Phi$ follows (Himmelberg, 1975, Theorem 5.1).

Lemma A.8. *Suppose*

(a) $(\Omega, \mathcal{F})$ is a measurable space,

(b) $\{X_i \mid i \in I\}$ is a countable family of second-countable topological spaces, and

(c) for every $i \in I$, $\Gamma_i : \Omega \Rightarrow X_i$ is weakly measurable with respect to $\mathcal{F}$.

Then, $\Gamma : \Omega \Rightarrow X$ is weakly measurable with respect to $\mathcal{F}$, where $X = \prod_{i \in I} X_i$ and $\Gamma(\omega) = \prod_{i \in I} \Gamma_i(\omega)$.

Proof. It follows from (b) that X is second-countable. Thus, if E is open in X, then $E = \bigcup_{j \in J} E_j$ for some collection $\{E_j \mid j \in J\}$ drawn from the basis for X. It is easily seen that $\{\omega \in \Omega \mid \Gamma(\omega) \cap E \neq \emptyset\} = \bigcup_{j \in J}\{\omega \in \Omega \mid \Gamma(\omega) \cap E_j \neq \emptyset\}$. It follows from this formula and the countability of J that it is sufficient to show that $\{\omega \in \Omega \mid \Gamma(\omega) \cap E \neq \emptyset\} \in \mathcal{F}$ for every E in the basis for X.

If E is in the basis for X, then $E = \prod_{i \in I} E_i$, where E_i is open in X_i for every $i \in I$. It is easy to check that $\{\omega \in \Omega \mid \Gamma(\omega) \cap E \neq \emptyset\} = \bigcap_{i \in I}\{\omega \in \Omega \mid \Gamma_i(\omega) \cap E_i \neq \emptyset\}$. As each Γ_i is weakly measurable with respect to $\mathcal{F}$, $\{\omega \in \Omega \mid \Gamma_i(\omega) \cap E_i \neq \emptyset\} \in \mathcal{F}$ for every $i \in I$. As I is countable, this implies $\{\omega \in \Omega \mid \Gamma(\omega) \cap E \neq \emptyset\} \in \mathcal{F}$. ∎

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