CDE
May, 1999

Centre for Development Economics

Collaboration and Competition in Networks

Dr. Sumit Joshi*
&
Dr. Sanjeev Goyal**

Working Paper No. 64

ACKNOWLEDGEMENTS

We are grateful to the participants at the Erasmus Theory Workshop and seminar series of Delhi School of Economics and Indian Statistical Institute for useful comments. Sumit Joshi would also like to thank Tinbergen institute, Erasmus University, and the Centre for Development Economics, Delhi School of Economics, for making available their resources to work on this project.

ABSTRACT

In a Cournot oligopoly, prior to choosing quantity, each firm has an opportunity to form *pair-wise* collaborative links with other firms. These pair-wise links lower costs of production of the firms which form a link and, if there are knowledge spillovers, also lower costs of other firms which are connected to them. The collection of pair-wise links defines a collaboration network. We characterize stable networks and compare them with efficient networks.

We find that the complete network is stable, irrespective of the assumptions on spillovers across collaboration links. This is in contrast to some recent work on group formation in cournot oligopoly, see e.g., Bloch (1995). Our finding is also interesting from a welfare point of view since, in such settings, the complete network is efficient from a social point of view.

* Department of Economics, 624 Funger Hall, George Washington University, 2201 G Street N.W., Washington D.C. 20052, USA. E-mail: sumjos@gwis2.circ.gwu.edu

** Econometric Institute, Erasmus University, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands. E-mail: goyal@few.eur.nl

1 Introduction

Consider a set of firms in an oligopolistic industry which compete in the market by setting quantities. Prior to this choice of quantities, they have an opportunity to form pair-wise collaborative links. We allow a firm to be a member of any number of distinct pair-wise collaborations. Thus in principle networks may involve intransitive collaborative relations: given distinct firms, i, j and k, a link between i and j and between j and k does not imply that i and k are also linked. The set of all pairwise collaborations defines a collaboration network. We study the nature of stable collaboration networks and compare them to the efficient networks.

We start with an analysis of the case where collaborations have *no knowledge spillovers*:[1] the lowering of costs for firms i and j from a collaboration with each other are independent of the number of other collaborations they individually have with other firms. We focus on the case where demand is linear and a firm's marginal cost decreases *linearly* with the number of collaborative links it has with other firms. We find that the *complete network, i.e., a network in which there is a collaborative link between every pair of firms, is both the unique stable network as well as the unique efficient network.* Thus we observe no conflict between stability and efficiency in this framework.

We then consider the case of *perfect spillovers*: when firms i and j establish a collaborative link, then they also benefit (in terms of cost reduction) by the same amount from the collaborative links each maintains with other firms. We observe first that *the complete network continues to be stable as well as efficient, in this setting.* It is, however, no longer the unique network to satisfy these properties. The set of stable networks is quite large: a stable network is either connected or has at most two components. We also find that any connected network in which every link is non-critical is stable. In particular, we find networks such as the wheel, where collaborative relations are non-transitive, are stable. If a stable network is disconnected then we show that the two components must be of unequal size.

We show that *a network is efficient if and only if it is connected.* Thus in addition to the complete network, structures such as the star/hub-spokes network and the line network are also efficient in this framework. A comparison of efficient and stable networks reveals that while there exist networks – such as the complete network and the wheel – which are efficient as well as stable, the two sets do not coincide. First, we observe that there exist stable

[1] Please refer to section 2 for formal definitions.

networks which are not efficient. An example is a network with N (where $N \geq 5$) firms in which $N-1$ firms form a complete component while a single firm is isolated. It can be shown that such a network is stable; however, it is inefficient since it is not a connected network. Second, we observe that there exist efficient networks which are not stable. One instance of such networks is the set of *minimally connected* networks.[2] Since minimally connected networks are connected, such networks are efficient. However, since all collaborative links in a minimally connected network are critical, they are not stable.

We also study the robustness of our results on stability and efficiency by allowing for general payoff functions, imperfect spillovers and asymmetric firms. In the first extension, we focus on the model of no-spillovers. We maintain the assumption that marginal costs are strictly decreasing in the number of direct collaborative link that a firm has with other firms; we do not require any more that the reduction in marginal cost is linear in the number of links. We first show that *every stable graph has the transitivity property: if firms i and firm j have a collaboration and firm j has a collaboration with some other firm k, then it must be the case that firms i and k also have a collaboration.* In other words, if a stable graph consists of more than one component, then each component is complete. We then examine the size and number of components in a stable network. We show that components in a stable network must be of unequal size and derive bounds on the number of components in a stable network. Finally, we show that the complete graph is always stable. We then identify a class of situations in which it is, in fact, the only stable graph. The case of linearly decreasing marginal costs falls within this class.

We conclude with a discussion of the case of *imperfect spillovers* and the case of asymmetric firms. In the former case, the reduction in marginal costs falls exponentially as the links become more distant. The imperfect spillover case is difficult to characterize fully even in relatively simpler models where link formation is one-sided (for instance, Bala and Goyal (1998)) or where link formation is not rival (for instance, Jackson and Wolinsky (1996)). We offer some partial results on stable networks and demonstrate that the complete network continues to be stable.

Our paper is a contribution to the study of group formation and cooperation in oligopolies. In recent years, considerable work has been done on this subject; see e.g., Bloch (1995,1996), Ray and Vohra (1997), Yi (1996,1997) and Yi and Shin (1995). In this literature, group formation is modelled in terms of a *coalition structure* which is a partition of the set of firms. Each firm,therefore, can belong to one and only one element of the partition, referred to as

[2] A network is said to be minimally connected if deleting a collaborative link between any two firms results in a network which is not connected. Minimally connected networks include the star and the line networks.

a *coalition.* Moreover, within a coalition, every firm is assumed to be symmetrically located, and thus avails of similar rewards from membership. Putting it differently, it is the number of firms in each coalition which determines of profits of each firm.

In contrast, we develop a model of group formation in an oligopoly based on pair-wise links between firms. This framework is richer than that of coalition structures in an important aspect: it accommodates collaborative relations that are *non-transitive.* The existing work on coalition structures basically restricts attention to collaboration structures are transitive. This restriction can be motivated quite well in some cases such as cartel formation, but it appears to be unduely restrictive for studying the nature of research and development joint ventures between firms. Thus, for example, in a three firm world, it is possible that firm 1 has a pair-wise link with 2 and 3, respectively, while the latter two firms have no pair-wise collaboration with each other. In this network, if the spillovers across pair-wise links are imperfect or absent, then the profit of firm 1 on the one hand, and firms 2 and 3 on the other hand, will be different.

Our approach is inspired by the recent work on strategic models of network formation; see e.g., Bala and Goyal (1998), Goyal (1993), Jackson and Wolinsky (1996), and Krannert and Minehart (1998). In the context of this literature, the primary contribution of the present paper is to show how ideas and concepts of network formation can be applied to a concrete problem in the theory of industrial organization. In particular, in our model, the formation of collaborative links generates benefits which are *rival*: when firms i and j form a collaborative link, they lower their marginal costs which in turn always lowers the profits of those firms who are not directly or indirectly linked to i and j[3]. Thus we study an example of network formation characterized by *negative externalities.* To the best of our knowledge this is the first paper to examine the architecture of collaboration networks in such a setting.

We now contrast our results with those in the existing literature. Bloch (1995,1996) develops a *sequential coalition unanimity* game in which firms propose coalitions and a coalition is formed only if every member of a proposed coalition agrees to become a member. Each firm's marginal cost is linearly declining in the size of the coalition of which it is a member. After coalitions are formed, the firms compete as Cournot oligopolists in a market with linear homogeneous demand. Bloch demonstrates that there is a unique stable coalition structure, in which firms are divided into two unequal groups. In particular, with N firms, the number of firms in the larger coalition is the integer closest to $3(N+1)/4$. The efficient coalition structure, however, is the grand coalition which cannot be achieved as a stable outcome of

[3]In the presence of spillovers, the profits of firms linked directly or indirectly (through a chain of intermediate links) to i and j may also increase.

the game. By contrast, we find in the context of the linear model that, *if spillovers are absent*, then the complete network/grand coalition is the unique stable as well as the unique efficient network. In the presence of perfect spillovers, we find that the complete network is still stable and efficient. However, other networks also arise in this case. In particular, we find that if the stable network is diconnected then it consists of two components of unequal size. This last result is similar to the findings of Bloch.

Yi and Shin (1995) propose a *simultaneous open membership* game in which all players announce their decision to form coalitions at the same time and non-members cannot be excluded from joining a coalition. They obtain the grand coalition as the stable outcome of the open membership game. Their approach is akin to a game in which the decision to join a coalition is one-sided. In such a game, and in the presence of perfect spillovers, a member of a smaller group always has an incentive to join a larger group. In our paper, by contrast, link formation is based on pair-wise incentive compatibility, and it is therefore interesting to observe that a grand coalition can be obtained in this setting also. Thus our result provides an alternative explanation as to how a grand coalition may emerge.

Ray and Vohra (1997) propose a *sequential equilibrium binding agreements* game, in which a sub-coalition is permitted to block a given strategy vector, thereby making the coalition structure finer. However, the blocking coalition does not naively take the strategy of the complement as given. Rather, it looks forward to an equilibrium that will be induced as the complementary members of the coalition structure react to its blocking. An equilibrium binding agreement is one which cannot be blocked by any coalition in a given coalition structure. Ray and Vohra apply this idea to a Cournot oligopoly with linear homogeneous demand and constant marginal cost. In their framework, marginal costs are not affected by coalition formation (the focus is on collusion in the quantity game). Considering just aggregate profits, they show that the grand coalition is efficient. They then demonstrate a cyclical pattern of stability as the number of players increase. In particular, while the grand coalition can be supported by an equilibrium binding agreement for very low and very high values of N, it is not immune to blocking by some coalition for intermediate values of N. In contrast, in a model where marginal costs are sensitive to link formation, we obtain the complete network as the unique stable and efficient network, independent of the value of N.

The model is presented in section 2; the analysis with no spillovers is presented in section 3 while the analysis with perfect spillovers is presented in section 4. Section 5 present an extension to general marginal costs with no spillovers. Section 6 concludes with a discussion of the imperfect spillover case and the case of asymmetric firms.

2 Basic Model

We consider a setting in which a set of firms first choose their collaboration links with other firms. These collaboration agreements are pair-wise and help lower marginal costs of production. The firms then compete in a market by choosing quantities. We are interested in the network of collaboration that emerges in this setting. We now develop the required terminology and provide some definitions.

2.1 Preliminaries

Let $\mathcal{N} = \{1, 2, ..., N\}$ denote a finite set of firms. To avoid trivialities, we shall assume that $N \geq 3$. For any $i, j \in \mathcal{N}$, the pair-wise relationship between the two firms is captured by a binary variable, $g_{ij} \in \{0, 1\}$; $g_{ij} = 1$ means that a direct link (joint venture) is established between firms i and j while $g_{ij} = 0$ means that no direct link is formed. By definition, $g_{ii} = 1 \ \forall i \in \mathcal{N}$. A *network*, $g = \{(g_{ij})_{i,j \in \mathcal{N}}\}$, is a formal description of the pair-wise collaboration relationships that exist between the firms in $\mathcal{N}$. We let $\mathcal{G}$ denote the set of all networks. Two special cases are the complete network, g^c, in which $g_{ij} = 1 \ \forall i, j \in \mathcal{N}$, and the empty network, g^e, in which $g_{ij} = 0 \ \forall i, j \in \mathcal{N}$, $i \neq j$. Let $g + g_{ij}$ denote the network obtained by replacing $g_{ij} = 0$ in network g by $g_{ij} = 1$. Similarly, let $g - g_{ij}$ denote the network obtained by replacing $g_{ij} = 1$ in network g by $g_{ij} = 0$.

Let $N(g) = \{i \in \mathcal{N} : \exists j \neq i \ s.t. \ g_{ij} = 1\}$. Each firm in $N(g)$ has at least one direct link to another distinct firm in the network g. Therefore, $N(g^c) = \mathcal{N}$ and $N(g^e) = \emptyset$. We will let $|N(g)|$ denote the cardinality of $N(g)$. There exists a path between firms i and j, either if $g_{i,j} = 1$ or if there exists a distinct set of firms $\{i_1, \ldots, i_n\} \subset N(g)$ such that $g_{ii_1} = g_{i_1 i_2} = g_{i_2 i_3} = \cdots = g_{i_n j} = 1$. Given any two firms i and j, let $d_{ij}(g)$ denote the number of links in the shortest path between i and j in the network g. We shall use the convention that $d_{ij}(g) = \infty$ if there exists no path between i and j in g. We refer to $d_{ij}(g)$ as the geodesic distance between firms i and j in g. For instance, $d_{ij}(g^c) = 1$ and $d_{ij}(g^e) = \infty$ $\forall i, j \in \mathcal{N}$.

A network, $g' \subset g$, is a *component* of g if for all $i, j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j, and for all $i \in N(g')$ and $j \in N(g)$, $g_{ij} = 1$ implies $g_{ij} \in g'$. Generally, in a component g' with three or more agents, there will exist agents i and j such that $d_{ij}(g') \geq 2$. We shall say that a component $g' \subset g$ is *complete* if $g_{ij} = 1$ for all $i, j \in N(g')$.

2.2 Payoffs

Given a network g, a firm $i's$ cost function is specified as follows:

$$C_i(q_i, g) = c_i(g)q_i, \quad i \in \mathcal{N} \tag{1}$$

where q_i denotes the output of firm i. We suppose that a firm's marginal cost is decreasing in the number of direct collaboration links it has. It is possible that a firm's costs be influenced by the collaboration links of other firms; this depends on the nature of spillovers across links. We consider two cases: no spillovers and perfect spillovers. Let $\eta_i(g, 1)$ be the number of firms who have a direct link with firm i, in network g; likewise, let $\eta_i(g, k)$ be the number of firms for which $d_{ij} = k$ in network g. We focus on the case of linearly decreasing costs.[4] We first state the no spillovers case:

$$c_k(g) = c_k(\eta_k(g,1)) = \gamma_0 - \gamma\eta_i(g,1)\,,\ \gamma_0 > N\gamma > 0\,,\ i \in \mathcal{N} \tag{\textbf{A.1}}$$

We next present the case of perfect spillovers:

$$c_i(g) = c(\eta_i(g)) = \gamma_0 - \gamma\left[\sum_{k=1}^{N-1} \eta_i(g,k)\right],\ \gamma_0 > N\gamma > 0,\ i \in \mathcal{N} \tag{\textbf{A.1}'}$$

The two expressions above are natural interpretations in the network framework of the specification used in Bloch (1995), where marginal cost of firm i decreases linearly in the number of firms belonging to the same coalition as i. Bloch (1995) provides a number of examples which generate the above linear specification for marginal cost. The last equality sign in (A.1)-(A.1') reflects the assumption that the firms are symmetric, in an ex-ante sense.

In the second stage, the firms engage in Cournot competition in a homogeneous product market where they face a linear inverse demand function given by:

$$P = \alpha - \sum_{i \in \mathcal{N}} Q_i \tag{2}$$

Given any network g from the first stage, the second stage Cournot competition between firms yields the following equilibrium output:

$$Q_i(g) = \frac{\alpha - Nc_i(g) + \sum_{j \neq i} c_j(g)}{N+1}\,,\ i \in \mathcal{N} \tag{3}$$

[4]The more general case of cost reduction as well as imperfect spillovers are considered in section 5.

Aggregate Cournot output, given g, is:

$$Q(g) = \sum_{i \in \mathcal{N}} Q_i(g) = \sum_{i \in \mathcal{N}} \frac{\alpha - c_i(g)}{N+1} \tag{4}$$

The second stage Cournot profits for a firm, given g, are:

$$\Pi_i(g) = Q_i^2(g)\,,\ i \in \mathcal{N} \tag{5}$$

In our study of *stable* networks, we will find it convenient to use a positive monotonic transformation of the firm's profits to write the payoff as follows:

$$\pi_i(g) = \alpha - N c_i(g) + \sum_{j \neq i} c_j(g)\,,\ i \in \mathcal{N} \tag{6}$$

We assume some restrictions on the parameters to ensure that each firm produces a positive quantity in the Cournot game.

$$0 < c_i(g) \leq \bar{c} < \infty,\ \forall i \in N,\ \forall g \in \mathcal{G};\ \alpha > 3N\bar{c}. \tag{\textbf{A}.2}$$

2.3 Stable and Efficient Networks

We employ a relatively weak notion of stability which is based on the idea that while links are formed bilaterally, they can be severed unilaterally.[5] Formally, the network g is *stable* if for all $i, j \in \mathcal{N}$:

(i) $\pi_i(g) \geq \pi_i(g - g_{ij})$ and $\pi_j(g) \geq \pi_j(g - g_{ij})$

(ii) if $\pi_i(g + g_{ij}) > \pi_i(g)$, then $\pi_j(g + g_{ij}) < \pi_j(g)$

In words, in a stable network, any firm that is directly linked to another has no incentive to sever the link and any two firms that are not directly linked have no incentive to form a direct link with each other.

[5] We have borrowed this concept from Jackson and Wolinsky (1996).

In order to study *efficient* networks, we need to consider aggregate welfare. For any network g, this is defined as the sum of consumer surplus and aggregate profits of the N firms:

$$W(g) = \frac{1}{2}Q^2(g) + \sum_{i \in \mathcal{N}} Q_i^2(g) \tag{7}$$

We shall say that a network g^* is *efficient* if $W(g^*) \geq W(g)$, for all $g \in \mathcal{G}$

3 The Case of No Spillovers

In this section we consider the case where the marginal costs of a firm decrease in the number of firms with which it is directly linked, but are unaffected by indirect links.

Proposition 3.1 *Suppose (A.1)-(A.2) hold. Then the complete network, g^c, is the unique stable network.*

The intuition behind this result is as follows: if two firms form a link then given (A.1) the costs of all other firms are unaffected. The cost advantage to both firms is the same under (A.1). An inspection of the profit expression in (6) reveals that the positive effects on the profits of a firm i from a link with another firm j is given by $N\gamma$, while the negative effects are given by γ. Thus link formation is clearly profit enhancing. This argument shows that any network other than the complete network cannot be stable. To see why the complete network is stable note that no further links can be added, while the deletion of a link by a firm i, with (say) firm j only increases the costs of firm i and j but leaves the costs of all other firms unaffected, lowering profits of firm i by $(N-1)\gamma$. Thus it is not profitable to delete links either. This completes the argument.

Proposition 3.2 *Suppose that (A.1)-(A.2) hold. Then the complete network, g^c, is the unique efficient network.*

We use a general result on Cournot quantities in proving this result. This is stated as Lemma 3.1.

Lemma 3.1 *Suppose* A.1 *and* A.2 *hold. Let g' be any component of a network g. Then $\sum_{i \in N(g')} Q_i(g)$ is maximized when g' is complete. In particular, aggregate output, $Q(g)$ is a maximum for the complete network, g^c.*

Proof of Lemma 3.1: Consider a component g' in a network g that is not complete. Let $|N(g')| = n'$, $n' \leq N$. Consider firms $i, j \in N(g')$ such that $g_{ij} = 0$. Under (A.2), all firms produce a strictly positive output in a Cournot equilibrium.

$$\sum_{k \in N(g')} [Q_k(g + g_{ij}) - Q_k(g)]$$

$$= \sum_{k \in N(g')} \left[\frac{\alpha - Nc_k(g + g_{ij}) + \sum_{j \neq k} c_j(g_i + g_{ij})}{N+1} - \frac{\alpha - Nc_k(g) + \sum_{j \neq k} c_j(g_i)}{N+1} \right]$$

$$= \frac{1}{N+1} \sum_{k \in N(g') \setminus \{i,j\}} \left[N(c_k(g) - c_k(g + g_{ij})) + \sum_{j \neq k} c_j(g_i + g_{ij}) - \sum_{j \neq k} c_j(g_i) \right]$$
$$+ \frac{1}{N+1} \sum_{k=i,j} \left[N(c_k(g) - c_k(g + g_{ij})) + \sum_{l \neq k} c_l(g_i + g_{ij}) - \sum_{l \neq k} c_l(g_i) \right]$$

On rearranging terms and simplifying, we get:

$$= \frac{1}{N+1} \sum_{k=i,j} [c_k(g) - c_k(g + g_{ij})]$$

Under (A.1), the last expression is strictly positive. This completes the proof. $\triangle$

The proof of Proposition 3.1 proceeds by showing that the aggregate social welfare increases every time a pair of firms previously unconnected form a link. Thus we compare two networks, g and $g + g_{ij}$ and show that the welfare levels in the two networks satisfies the following relation: $W(g + g_{ij}) > W(g)$. It is relatively straightforward to show that the aggregate output and hence consumer surplus increases as costs of the two firms i and j go down. The behaviour of the second term in the welfare expression, the aggregate profits of the firms is, however, unclear. We exploit the linearity of the cost reduction in determining the sign and magnitude of that term. The details of the computations are given in the appendix.

4 The Case of Perfect Spillovers

We now consider the case where a firm's marginal cost decreases with the number of firms to whom it is directly as well as those to whom it is indirectly linked.[6] We start by showing that the complete network is pair-wise stable. This result also shows that the set of pair-wise stable networks is non-empty.

Proposition 4.1 *Suppose that (A.1')-(A.2) hold. Then g^c is pair-wise stable.*

Proof In g^c, the payoff to each firm is:

$$\pi_i(g^c) \;=\; \alpha - c_i(g^c) \;=\; \alpha - \gamma_0 + \gamma(N-1) \tag{8}$$

The payoff to firm i after deleting a link g_{ij} is given by:

$$\pi_i(g^c - g_{ij}) \;=\; \alpha - N c_i(g^c - g_{ij}) + c_j(g^c - g_{ij}) + \sum_{k \neq i,j} c_k(g^c - g_{ij}) \tag{9}$$

Under the assumption of perfect spillovers, $c_i(g^c - g_{ij}) = c_j(g^c - g_{ij}) = c_i(g^c) = \gamma_0 - \gamma(N-1)$ and $c_k(g^c - g_{ij}) = \gamma_0 - \gamma(N-1), k \neq i, j$. Therefore, $\pi_i(g^c) = \pi_i(g^c - g_{ij})$ and there is no incentive to delete the link g_{ij}. $\triangle$

We now study the nature of stable networks more generally. Recall that there exists a path between i and j if either $g_{ij} = 1$ or there is a sequence of firms $i_1, i_2, \ldots, i_n$ such that $g_{ii_1} = g_{i_1 i_2} = \cdots = g_{i_n j} = 1$. We say that a network is *connected* if for every pair of firms $i, j \in \mathcal{N}$, there exists a path between them. A network is *unconnected* if there exists some $i, j \in \mathcal{N}$ such that there is no path between them. A network is *minimally connected* if $g - g_{ij}$ is unconnected for each $g_{ij} \in g$, $g_{ij} = 1$. Examples of minimally connected networks include the star and the line network.

Our next result looks at connected networks. We will say that a link g_{ij} is *critical* in the network g if $g_{ij} = 1$ in g and there is no path between i and j in the network $g - g_{ij}$. Likewise, g_{ij} is *non-critical* in the network g if $g_{ij} = 1$ in g and there exists a path between i and j in the network $g - g_{ij}$.

[6] This corresponds to the original Myerson (1977) formulation where two players cooperate in any given network if they are directly connected by a collaborative link or indirectly by a path of intermediate collaborative links.

Proposition 4.2 *Suppose that (A.1')-(A.2) hold. If g is connected then g is stable if every link in g is non-critical. Moreover, if g is minimally connected, then it is not stable for $N > 3$.*

Proof: If g is connected, then clearly adding links makes no difference to the payoffs because it leaves the marginal costs of all firms unaffected. Moreover, deleting a single link g_{ij} again makes no difference since g_{ij} is non-critical. Thus, if g is connected and every link is non-critical, then g is stable.

Let g be a minimally connected network. In g, all firms have a marginal cost of $\gamma_0 - \gamma(N-1)$. By definition of minimal connectedness, there exists some $i, j \in \mathcal{N}$ such that in $g - g_{ij}$, firm j is isolated from the other $N-1$ firms and firm i is connected to $N-2$ firms other than firm j. The new marginal costs are $c_i(g - g_{ij}) = \gamma_0 - \gamma(N-2), c_j(g - g_{ij}) = \gamma_0$ and $c_k(g - g_{ij}) = \gamma_0 - \gamma(N-2), k \neq i, j$. The payoff to firm i before and after the deletion of the link are:

$$\pi_i(g) \;=\; \alpha - [\gamma_0 - \gamma(N-1)] \tag{10}$$

$$\pi_i(g - g_{ij}) \;=\; \alpha - N[\gamma_0 - \gamma(N-2)] + (N-2)[\gamma_0 - \gamma(N-2)] + \gamma_0 \tag{11}$$

It can now be verified that $\pi_i(g - g_{ij}) - \pi_i(g) = \gamma(N-3) > 0$ for $N > 3$. Therefore, g is not stable for $N > 3$. △

As a partial converse of the first part of Proposition 4.2 we have that if g is a stable connected network with $N > 3$ firms, then at least one link in g is non-critical. If not, then all links in g would be critical implying that g is minimally connected. But, from the second part of Proposition 4.2, this would contradict the stability of g.

We next look at the class of unconnected networks. We first show that components in a stable network must be of unequal size. The argument is as follows: suppose that there are two components 1 and 2, of size k each in a stable network. Then the gross profit advantage for a firm i in component 1 to form a link with another firm j in component 2, is given by $Nk\gamma$, while the disadvantage in terms of reduced costs of other firms is given by $(k^2 + (k-1)k)\gamma$. The former effect is larger if $Nk\gamma > (k^2 + (k-1)k)\gamma$, which is equivalent to the requirement that $N > 2k - 1$, which is always true. This argument is summarized in the following lemma:

Lemma 4.1 *Suppose that (A.1')-(A.2) hold. In a stable network the components are of unequal size.*

The next result derives an upper bound on the number of components.

Proposition 4.3 *Suppose that (A.1')-(A.2) hold. If g is a stable network, then it has at most two components.*

Proof: Suppose g is a stable network with more than two components. Let g' and g'' be the smallest two components with $|N(g')| = n'$ and $|N(g'')| = n''$. Then, by the definition of a component, $n', n'' < (N+1)/2$. Let $i \in N(g')$ and $j \in N(g'')$. Then, the payoff of firm i before and after forming a link with firm j are:

$$\pi_i(g) = \alpha - (N - n' + 1)[\gamma_0 - \gamma(n' - 1)] + n''[\gamma_0 - \gamma(n'' - 1)] + \sum_{k \notin N(g') \cup N(g'')} c_k(g) \quad (12)$$

$$\begin{aligned} \pi_i(g + g_{ij}) &= \alpha - (N - n' + 1)[\gamma_0 - \gamma(n' - 1 + n'')] \\ &\quad + n''[\gamma_0 - \gamma(n' + n'' - 1)] + \sum_{k \notin N(g') \cup N(g'')} c_k(g + g_{ij}) \end{aligned} \quad (13)$$

Subtracting (12) from (13) and simplifying:

$$\pi_i(g + g_{ij}) - \pi_i(g) = \gamma n'' [N - 2n' + 1] \quad (14)$$

From (14), $\pi_i(g + g_{ij}) > \pi_i(g)$ if and only if $n' < (N+1)/2$, which is true by hypothesis. An identical argument establishes that $\pi_j(g + g_{ij}) > \pi_j(g)$ if and only if $n'' < (N+1)/2$, which is also true by hypothesis. But this contradicts the stability of g. $\triangle$

We now give an example of a stable network with two components. Consider a network where one component contains $N-1$ firms while the other one contains the remaining single firm. Moreover, let the first component be complete. It can be verified that if $N \geq 4$ then this network is stable. In such a network, the isolated firm wishes to form a link with the large component, but none of the firms in the large component wish to form a link with the isolated firm.[7]

We study the nature of efficient networks next. In contrast to the no spillover case, the set of efficient networks is quite large.

Proposition 4.4 *Suppose that (A.1')-(A.2) hold. A network g is efficient if and only if it is connected.*

Proof Consider the "only if" part. Consider any network g with two or more components. Let g' and g'' be the two largest components where $|N(g')| = n' \leq n'' = |N(g'')|$. Let $i \in N(g')$ and $j \in N(g'')$. Consider the network $g+g_{ij}$. We will show that $W(g+g_{ij}) > W(g)$; thus any disconnected network is welfare dominated, implying that an efficient network must be connected. From expression (4) it follows that $Q(g + g_{ij}) > Q(g)$; therefore, consumer surplus strictly increases with the addition of the link $g_{ij} = 1$. To show that aggregate profits also increase, we need a few intermediate facts. First, comparing the outputs of firms before and after the formation of the link $g_{ij} = 1$, we observe that for all $h \in N(g')$, $l \in N(g'')$ and $k \notin N(g') \cup N(g'')$:

$$Q_l(g) \geq Q_h(g) \geq Q_k(g) \tag{15}$$

$$Q_l(g + g_{ij}) = Q_h(g + g_{ij}) > Q_k(g + g_{ij}) \tag{16}$$

Second, comparing the change in output for each firm before and after the formation of the link $g_{ij} = 1$:

$$Q_h(g + g_{ij}) - Q_h(g) = \frac{\gamma n''(N - 2n' + 1)}{(N + 1)}, \quad h \in N(g') \tag{17}$$

$$Q_l(g + g_{ij}) - Q_l(g) = \frac{\gamma n'(N - 2n'' + 1)}{(N + 1)}, \quad l \in N(g'') \tag{18}$$

[7] To see this note that, in the linear model with perfect spillovers, the advantage for a firm in the large component from forming a link is given by $N\gamma$, while the disadvantage, reflected in the falling costs of the other firms, is given by $(N-1)\gamma + (N-2)\gamma$. The latter is larger than the former if $N > 3$. By contrast, the advantage for the singleton firm from such a link is $N.(N-1)\gamma$, while the disadvantage is $(N-1)\gamma$. Clearly, the former is larger than the latter, so long as $N \geq 2$.

$$Q_k(g+g_{ij}) - Q_k(g) \;=\; -\frac{2\gamma n'n''}{(N+1)}, \;\; k \notin N(g') \cup N(g'') \tag{19}$$

Combining the above, we get:

$$\sum_{m\in\mathcal{N}} \left[Q_m^2(g+g_{ij}) - Q_m^2(g)\right] = \sum_{m\in\mathcal{N}} [Q_m(g) + Q_m(g+g_{ij})]\,[Q_m(g+g_{ij}) - Q_m(g)] \tag{20}$$

$$\geq \frac{2\gamma n'n''}{N+1}\,[Q_h(g) + Q_h(g+g_{ij})] > 0. \tag{21}$$

Details of the computations are provided in the appendix. This proves that g is welfare dominated by $g + g_{ij}$. Since g was an arbitrary disconnected network, this implies that any disconnected network is welfare dominated. Hence an efficienct network is necessarily connected.

Regarding the "if" part, suppose g is some connected network which is not efficient. Then there exists some $g^* \in \mathcal{G}$ such that $W(g^*) > W(g)$. If g^* is connected, then it yields the same aggregate welfare as g, contradicting $W(g^*) > W(g)$. Now suppose that g^* is unconnected and let $g^1, g^2, \ldots, g^K$ be its components such that $|N(g^1)| \leq |N(g^2)| \leq \ldots |N(g^K)|$. Letting i_n denote a firm in $|N(g^n)|$, $n = 1, 2, \ldots, K$, we can mimic the argument in the first part to show that $W(g^* + g_{i_{K-1}i_K}) > W(g^*)$. Repeating this argument $K-1$ times strictly increases aggregate welfare in each step and culminates with the connected network, $g^* + \sum_{n=1}^{K-1} g_{i_n i_{n+1}}$. By the assumption of perfect spillovers, $W(g) = W(g^* + \sum_{n=1}^{K-1} g_{i_n i_{n+1}}) > W(g^*)$ contradicting $W(g^*) > W(g)$. This proves the "if" part of the proposition. △

5 General costs with no spillovers

We once again consider the model where a firm's marginal cost decreases with the number of firms to whom it is directly linked but are unaffected by the number of indirect links to other firms. We specify marginal cost of firm i quite generally by $c_i(g) \equiv c(\eta_i(g,1))$. We assume that:

$$c(k) > c(k+1), \ \forall k \in \{1, \ldots, N-1\} \tag{A.3}$$

Our first result establishes that every stable network has the following transitivity property: if firms i and firm j have a collaboration and firm j has a collaboration with some other firm k, then it must be the case that firms i and k also have a collaboration.

Proposition 5.1 *Suppose (A.2)-(A.3) hold. A stable network consists of complete components.*

The proof uses two lemmas; these results are also of some independent interest and so we state and prove them in the text.

Lemma 5.1 *Consider a stable network, g. If $\eta_i(g,1) = \eta_j(g,1)$, then $g_{ij} = 1$.*

Proof: Let g be stable. If $\eta_i(g,1) = \eta_j(g,1) = N$, then by definition $g_{ij} = 1$. Therefore, consider the case where $\eta_i(g,1) = \eta_j(g,1) < N$ and $g_{ij} = 0$. The payoffs to i and j are identical if $\eta_i(g,1) = \eta_j(g,1)$. The payoff to firm i is given by:

$$\pi_i(g) = \alpha - Nc(\eta_i(g,1)) + c(\eta_j(g,1)) + \sum_{k \neq i,j} c(\eta_k(g,1)) \tag{22}$$

In the network, $g + g_{ij}$, the payoff to firm i is given by:

$$\pi_i(g + g_{ij}) = \alpha - Nc(\eta_i(g,1)+1) + c(\eta_j(g,1)+1) + \sum_{k \neq i,j} c(\eta_k(g+g_{ij},1)) \tag{23}$$

Note that $\eta_k(g + g_{ij}, 1) = \eta_k(g,1)$ for $k \neq i, j$. Therefore, $c(\eta_k(g+g_{ij},1)) = c(\eta_k(g,1))$. On rearranging terms, it follows that $\pi_i(g+g_{ij}) > \pi_i(g)$ if and only if $c(\eta_i(g,1)+1) < c(\eta_i(g,1))$ which is true by (A.3). But this contradicts the hypothesis that g is stable. $\triangle$

Lemma 5.1 has some interesting implications for the nature of stable networks. The first implication is that the empty network, g^e, is not stable. The second implication is that a stable network cannot have two or more singleton components. The third implication is that the star/hub-spokes network [8] is not stable. This is because in all these networks, there are

[8] A star network has one central agent i, with $g_{ij} = 1$ for every $j \neq i$ and $g_{jk} = 0$ for every $j, k \neq i$.

at least two firms i and j who have the same number of direct links but $g_{ij} = 0$. By Lemma 5.1, these two firms have an incentive to form a direct link.

Lemma 5.2 *Consider a stable network g. If firms i and j belong to a component of g, then $g_{ij} = 1$.*

Proof: Consider a stable network g. Suppose that g has a component g' where $|N(g')| = k \geq 3$ (for $k = 1$ or $k = 2$, the claim is obviously true). Let there be $l \leq k$ agents in g' with $\eta_i(g', 1) = k$. If $l = k$, then we are done. Therefore, suppose that $l < k$. Since l agents have $\eta_i(g', 1) = k$, it follows that for any $j \in N(g')$, $\eta_j(g', 1) \geq l+1$. Moreover, since only l agents have $\eta_i(g', 1) = k$, it follows that there are $k - l$ agents with $l + 1 \leq \eta_i(g', 1) \leq k - 1$. Hence $\eta_i(g', 1)$ can take on $(k - l - 1)$ possible values. Since there are $k - l$ firms, it follows that there exist at least two firms i and j with $\eta_i(g', 1) = \eta_j(g', 1) < k$. An application of Lemma 5.1 now implies that g is not stable, a contradiction. This completes the proof. $\triangle$

Combining Lemmas 5.1 and 5.2 yields us Proposition 5.1. Moreover, an implication of Lemma 5.1 is that if a stable network contains many components then they must be of unequal size.

The above results leave open the issue of existence of stable networks. The next result shows that the set of stable networks is non-empty.

Proposition 5.2 *Suppose (A.2)-(A.3) hold. Then the complete network, g^c, is stable.*

Proof: In $g^c, \eta_i(g^c, 1) = N - 1$, $\forall i \in \mathcal{N}$. Therefore, firm i has a marginal cost of $c(N - 1)$ and payoff of:

$$\pi_i(g^c) \;=\; \alpha - c(N-1) \tag{24}$$

There are no links to add so condition (ii) of stability is automatically satisfied. We check condition (i) next. Suppose we set $g_{ij} = 0$ for some pair i and j. In the ensuing network, $g^c - g_{ij}$, the payoff to i is given by:

$$\pi_i(g^c - g_{ij}) \;=\; \alpha - Nc(N-2) + c(N-2) + (N-2)c(N-1) \tag{25}$$

The payoff to firm j is identical. There is no incentive to delete link $g_{ij} = 1$ if $\pi_i(g^c - g_{ij}) < \pi_i(g^c)$. But this is equivalent to $c(N - 1) < c(N - 2)$ which is true by (A.3). $\triangle$

We would like to further characterize the structure of networks. A first step in that direction is the next result. It provides a complete characterization for a restricted set of parameters.

Proposition 5.3 *Suppose (A.2)-(A.3) hold and in addition* $N[c(k)-c(k+1)] > c(l)-c(l+1)\ \forall l,k \in \{1,\ldots,N-1\}$. *Then* g^c *is the unique stable network.*

Proof: Proposition 5.2 shows that under (A.2)-(A.3), g^c is stable. We now show that under the hypothesis, $N[c(k)-c(k+1)] > c(l)-c(l+1)\ \forall l,k \in \{1,\ldots,N-1\}$, $g \neq g^c$ is not stable, thus establishing that the complete network is the unique stable network. Consider some g with $g_{ij} = 0$. We show that both i and j are strictly better off by forming a link. The payoff to i in network g is given by:

$$\pi_i(g) \;=\; \alpha - Nc(\eta_i(g,1)) + c(\eta_j(g,1)) + \sum_{k\neq i,j} c(\eta_k(g,1)) \tag{26}$$

In the network, $g+g_{ij}$, the payoff to firm i is given by:

$$\pi_i(g+g_{ij}) \;=\; \alpha - Nc(\eta_i(g,1)+1) + c(\eta_j(g,1)+1) + \sum_{k\neq i,j} c(\eta_k(g+g_{ij},1)) \tag{27}$$

Since $\eta_k(g+g_{ij},1) = \eta_k(g,1)$, it follows that $c(\eta_k(g+g_{ij},1)) = c(\eta_k(g,1))$. Comparing (26) and (27), we find that $\pi_i(g+g_{ij}) > \pi_i(g)$ if and only if:

$$N[c(\eta_i(g,1)) - c(\eta_i(g,1)+1)] \;>\; c(\eta_j(g,1)) - c(\eta_j(g,1)+1) \tag{28}$$

Similarly for firm j, $\pi_j(g+g_{ij}) > \pi_j(g)$ if and only if:

$$N[c(\eta_j(g,1)) - c(\eta_j(g,1)+1)] \;>\; c(\eta_i(g,1)) - c(\eta_i(g,1)+1) \tag{29}$$

Inequalities (28) and (29) are satisfied if $N[c(k)-c(k+1)] > c(l)-c(l+1)$, for all $l,k \in \{1,2,...,N-1\}$. Thus, g is not stable, a contradiction. $\triangle$

We now examine the number of components that can arise in stable networks under more general conditions. Let k be the total number of firms to which a given firm is directly linked. We examine the cases where the marginal cost function is concave and convex, respectively, in k. It is easily seen that if the function is concave then it displays increasing differences, i.e., $c(k) - c(k+1) > c(k-1) - c(k)$, for all $k \in \{1, \cdots, N-1\}$.[9]. Likewise, convexity implies decreasing differences, i.e., $c(k) - c(k+1) < c(k-1) - c(k)$, for all $k \in \{1, \cdots, N-1\}$

Proposition 5.4 *Suppose (A.2)-(A.3) hold and the marginal cost function has the property of decreasing differences in k. Let* $z \in \{1, \cdots, N-1\}$ *be the smallest integer for which* $N[c(z) - c(z+1)] \leq c(0) - c(1)$. *Then, in any stable network, there is at most one component of size smaller than z. If there is o z satisfying the above inequality, then the complete network is the unique stable network.*

Proof: If there is no z satisfying the conditions of the proposition, then the proof of Proposition 5.3 applies. We, therefore, focus on the case where such a z exists. We first show that given any two components, g' and g'' of g with $|N(g')| = z'$ and $|N(g'')| = z''$, $z' < z''$, each firm $j \in N(g')$, has an incentive to form a link with a firm $i \in N(g'')$. The payoff to $j \in N(g')$ is given by:

$$\pi_j(g) = \alpha - Nc(z') + c(z'') + \sum_{l \neq i,j} c(\eta_l(g, 1)) \tag{30}$$

The payoff to firm $j \in N(g')$, after adding a link with $i \in N(g'')$, is given by:

$$\pi_j(g + g_{ij}) = \alpha - Nc(z'+1) + c(z''+1) + \sum_{l \neq i,j} c(\eta_l(g + g_{ij}, 1)) \tag{31}$$

Note that $c(\eta_l(g, 1)) = c(\eta_l(g + g_{ij}, 1))$ for $l \neq i, j$. Therefore, $\pi_j(g + g_{ij}) > \pi_j(g)$ if and only if $N[c(z') - c(z'+1)] > c(z'') - c(z''+1)$. But this is true since $z' < z''$ and the property of decreasing differences is satisfied.

We next show that if z is any integer satisfying $N[c(z) - c(z+1)] \leq c(0) - c(1)$, then there is at most one component with less than z firms. Suppose not. Let there be two components g' and g'', with $N(g') = z'$ and $N(g'') = z''$ firms. Assume that $0 < z' < z'' < z$. We have already seen that every $j \in N(g')$ has an incentive to establish a link with any firm $i \in N(g'')$. Therefore, stability dictates that every firm $i \in N(g'')$ should have an incentive to *not* form a link with any firm $j \in N(g')$. In other words, it must be true that

[9] To see this note that by the definition of concavity, for any integer $1 < k < N$, $c(k) = c(\frac{1}{2}(k-1) + \frac{1}{2}(k+1)) > \frac{1}{2}c(k-1) + \frac{1}{2}c(k+1)$. Rearranging the above yields the property of increasing differences: $c(k) - c(k+1) > c(k-1) - c(k)$

$\pi_i(g) > \pi_i(g+g_{ij})$. Following the same argument above, this is equivalent to the requirement that $N[c(z'') - c(z''+1)] < c(z') - c(z'+1)$. However, $z'' < z$, and so by definition of z and the property of decreasing differences it follows that:

$$N[c(z'') - c(z''+1)] > c(0) - c(1) > c(z') - c(z'+1)$$

This contradicts the definition of stability of g and completes the proof. △

This result gives us an upper bound on the number of components. To illustrate this, let the oligopolistic industry consist of $N = 100$ firms and suppose that the parameters are such that $z = 30$. The above result tells us that at most one component can be of size smaller than 30; the smallest such component consists of a single firm. The remaining 99 firms have to be each of size 30 or more. Thus there can be at most 3 other components. Putting together these observations we get that there can be at most 4 components in any stable network. More generally, letting $[x]$ denote the smallest integer exceeding a real number x, we can say that if N is the number of firms and z is as defined in Proposition 5.4, then the number of components in a stable network is bounded above by $[(N-1)/z]$.

It may also be noted that an argument identical to that in Proposition 3.2, which exploits the assumption of no spillovers, can be constructed to show that the complete network is uniquely efficient in $\mathcal{G}$.

6 Conclusion

So far we have studied the nature of network formation with no spillovers and perfect spillovers and the case where all firms are ex-ante identical. We conclude with a brief discussion of the nature of network formation when spillovers are positive but imperfect or when firms are not ex-ante identical.

To look at imperfect spillovers, we shall study a linear specification of the costs. Suppose that $c_i(g)$, $i \in \mathcal{N}$ is defined as follows:

$$c_i(g) = \gamma_0 - \gamma[\eta_i(g,1) + \delta\eta_i(g,2) + \delta^2\eta_i(g,3) + \cdots + \delta^{N-2}\eta_i(g,N-1)] \qquad \textbf{(A.4)}$$

where $\eta_i(g,k)$ is the number of firms who are at a geodesic distance k from i and $0 < \delta < 1$ is the spillover parameter. Note that if $\delta = 0$, then we are back in the model with no spillovers, while $\delta = 1$ corresponds to the case of perfect spillovers. An argument along the lines of Proposition 4.1 shows that the complete network g^c continues to be stable in this setting. This indicates that the set of stable networks is non-empty. It can also be shown that networks in which two firms, i and j have the same number of direct and indirect links with identical firms $k \neq i, j$ but $g_{ij} = 0$ cannot be stable. This once again rules out networks with two or more singleton components, and the star network from being stable.

We saw in the no spillover case that stable networks generated a collaboration relationship that was transitive. Allowing firms to be asymmetric, while maintaining the assumption of no spillovers, generates stable networks where collaboration relations may be intransitive. For instance, let $N = 3$ and consider the following marginal costs for the three firms:

$$c_1 = \gamma + \beta\,, \qquad c_2 = c_3 = \gamma + 2\beta$$

where $\gamma, \beta > 0$. Every direct link reduces marginal cost by β unless costs are already at a minimum in which case links have no effects. The minimum marginal cost is γ. It can be easily verified that a star network, with either firm 2 or firm 3 as the center of the star, is uniquely stable.

A full characterization of network formation with imperfect spillovers and with asymmetric firms is one of the objectives of future research.

7 Appendix

Proof of Proposition 3.1: We first show that g^c is stable. In $g^c, \eta_i(g^c, 1) = N-1, \forall i \in \mathcal{N}$. Therefore, firm i has a marginal cost of $\gamma_0 - \gamma(N-1)$ and payoff of:

$$\pi_i(g^c) = (\alpha - \gamma_0) + \gamma(N-1) \tag{32}$$

There are no links to add so condition (ii) of stability is automatically satisfied. We check condition (i) next. Suppose we set $g_{ij} = 0$ for some pair i and j. In the ensuing network, $g^c - g_{ij}$, the payoff to i is given by:

$$\pi_i(g^c - g_{ij}) = \alpha - (N-1)[\gamma_0 - \gamma(N-2)] + (N-2)[\gamma_0 - \gamma(N-1)] = (\alpha - \gamma_0) \tag{33}$$

The payoff to firm j is identical. There is no incentive to delete link $g_{ij} = 1$ since $\pi_i(g^c) - \pi_i(g^c - g_{ij}) = \gamma(N-1) > 0$.

We now show that g^c is the unique stable network. Consider a stable network $g \neq g^c$. Then, there exists $i, j \in \mathcal{N}$ with $g_{ij} = 0$. We show that both i and j are strictly better off by forming a link. The payoff to i in the network g is given by:

$$\pi_i(g) = \alpha - Nc(\eta_i(g,1)) + c(\eta_j(g,1)) + \sum_{k \neq i,j} c(\eta_k(g,1)) \tag{34}$$

In the network, $g + g_{ij}$, the payoff to firm i is given by:

$$\pi_i(g + g_{ij}) = \alpha - Nc(\eta_i(g,1)+1) + c(\eta_j(g,1)+1) + \sum_{k/i,j \in} c(\eta_k(g + g_{ij}, 1)) \tag{35}$$

Since
$eta_k(g + g_{ij}, 1) = \eta_k(g,1)$, it follows that $c(\eta_k(g + g_{ij}, 1)) = c(\eta_k(g,1))$. Using (34) and (35) and the specification (A.1), we find that $\pi_i(g + g_{ij}) - \pi_i(g) = \gamma(N-1) > 0$. An identical argument establishes that for firm j, $\pi_j(g + g_{ij}) - \pi_j(g) = \gamma(N-1) > 0$. Thus, condition (ii) is violated and g is not stable, a contradiction. $\triangle$

Proof of Proposition 3.2: Consider any network $g \neq g^c$. Then, for some $i, j \in \mathcal{N}$, $g_{ij} = 0$. We now show that $g + g_{ij}$ yields strictly greater aggregate welfare than g. First, consider consumer surplus. Under (A.1), note that:

$$Q(g+g_{ij}) - Q(g) = \frac{2\gamma}{(N+1)}\,, \quad Q(g+g_{ij}) + Q(g) = 2\left[Q(g) + \frac{\gamma}{(N+1)}\right]$$

Therefore, the change in consumer surplus becomes:

$$\frac{1}{2}\left[Q^2(g+g_{ij}) - Q^2(g)\right] \;=\; \left[Q(g) + \frac{\gamma}{(N+1)}\right]\frac{2\gamma}{(N+1)} \tag{36}$$

Next, consider aggregate profits. For firm i (as well as for firm j):

$$Q_i(g+g_{ij}) - Q_i(g) = \frac{\gamma(N-1)}{(N+1)}\,, \quad Q_i(g+g_{ij}) + Q_i(g) = 2Q_i(g) + \frac{\gamma(N-1)}{(N+1)}$$

Therefore, the change in aggregate profits for firms i and j are:

$$\sum_{h=i,j}\left[Q_h^2(g+g_{ij}) - Q_h^2(g)\right] \;=\; \sum_{h=i,j}\left[2Q_h(g) + \frac{\gamma(N-1)}{(N+1)}\right]\left[\frac{\gamma(N-1)}{(N+1)}\right] \tag{37}$$

For firms $k \neq i, j$:

$$Q_k(g+g_{ij}) - Q_k(g) = \frac{-2\gamma}{(N+1)}\,, \quad Q_k(g+g_{ij}) + Q_k(g) = 2Q_k(g) - \frac{2\gamma}{(N+1)}$$

Therefore, the change in aggregate profits for firms $k \neq i, j$ are:

$$\sum_{k\neq i,j}\left[Q_k^2(g+g_{ij}) - Q_k^2(g)\right] \;=\; \sum_{k\neq i,j}\left[2Q_k(g) - \frac{2\gamma}{(N+1)}\right]\left[-\frac{2\gamma}{(N+1)}\right] \tag{38}$$

Summing up (36), (37) and (38), the change in aggregate welfare can be simplified to:

$$W(g+g_{ij}) - W(g) \;=\; \left[N\sum_{h=i,j} Q_h(g) - \sum_{k\neq i,j} Q_k(g)\right]\left[\frac{2\gamma}{(N+1)}\right] + \frac{2\gamma^2(N^2-2)}{(N+1)^2} \tag{39}$$

From (39), $W(g + g_{ij}) > W(g)$, if for the network g:

$$N \sum_{h=i,j} Q_h(g) - \sum_{k \neq i,j} Q_k(g) > 0. \tag{40}$$

To see that this is true we proceed as follows: First we check by direct computation that this inequality is satisfied for $N = 3$. Then we consider the case of $N \geq 4$. We suppose that g is a network in which there is a link between (say) i and some other agent $k \in N \backslash \{i, j\}$. (If not, then proceed to the next step.) Define a new network $g - g_{i,k}$. Straightforward but tedious calculations reveal that:

$$N \sum_{h=i,j} Q_h(g) - \sum_{k \neq i,j} Q_k(g) \geq N \sum_{h=i,j} Q_h(g - g_{i,j}) - \sum_{k \neq i,j} Q_k(g - g_{i,j}) \tag{41}$$

Proceeding inductively, we show that

$$N \sum_{h=i,j} Q_h(g) - \sum_{k \neq i,j} Q_k(g) \geq N \sum_{h=i,j} Q_h(g') - \sum_{k \neq i,j} Q_k(g') \tag{42}$$

where g' is defined as the network obtained from g after all links involving firms i and j are deleted.

Now consider the network formed when every pair of firms $k, l \in N \backslash \{i, j\}$ has a direct link and the firms i and j remain isolated singletons. Denote this new network by $\hat{g}$. It follows from Lemma 3.1 that the output of firms in $N \backslash \{i, j\}$ is (weakly) greater in $\hat{g}$ as compared to g'. Since the costs of firms i and j have remained the same, while the costs of all the other firms have (weakly) decreased, it folows from (3) that the combined output of firms i and j has (weakly) decreased in the move from g' to $\hat{g}$. These observations along with (42) allow us to state:

$$N \sum_{h=i,j} Q_h(g) - \sum_{k \neq i,j} Q_k(g) \geq N \sum_{h=i,j} Q_h(\hat{g}) - \sum_{k \neq i,j} Q_k(\hat{g}) \tag{43}$$

Thus to verify inequality (40), it suffices to check if the R.H.S. of (43) is positive. In $\hat{g}$:

$$Q_h(\hat{g}) = \frac{(\alpha - \gamma_0) - (N-2)(N-3)\gamma}{(N+1)}, \; h = i, j \tag{44}$$

$$Q_k(\hat{g}) = \frac{(\alpha - \gamma_0) + 3\gamma(N-3)}{(N+1)}, \ k \neq i,j \tag{45}$$

Substituting (44) and (45) into (40) and simplifying, aggregate welfare will increase if $(N+2)(\alpha-\gamma_0) > \gamma(2N+3)(N-2)(N-3) = \gamma[2N^3 - 7N^2 - 3N + 18]$. But this is true under our parametric restrictions (A.2) and (A.1) since $(N+2)(\alpha-\gamma_0) > (N+2)(2N-1)\gamma_0 > (N+2)(2N-1)N\gamma = [2N^3+3N^2-2N]\gamma > [2N^3-7N^2-3N+18]\gamma$. Therefore, starting from any $g \neq g^c$, adding an additional link strictly increases aggregate welfare. Hence aggregate welfare is maximized at g^c. $\triangle$

Computations in Proposition 4.4:

$$\sum_{m\in\mathcal{N}} \left[Q_m^2(g+g_{ij}) - Q_m^2(g)\right] = \sum_{m\in\mathcal{N}} \left[Q_m(g) + Q_m(g+g_{ij})\right]\left[Q_m(g+g_{ij}) - Q_m(g)\right]$$

$$\begin{aligned}
&= \sum_{h\in N(g')} \left[Q_h(g) + Q_h(g+g_{ij})\right]\left[Q_h(g+g_{ij}) - Q_h(g)\right] \\
&\quad + \sum_{l\in N(g'')} \left[Q_l(g) + Q_l(g+g_{ij})\right]\left[Q_l(g+g_{ij}) - Q_l(g)\right] \\
&\quad + \sum_{k\notin N(g')\cup N(g'')} \left[Q_k(g) + Q_k(g+g_{ij})\right]\left[Q_k(g+g_{ij}) - Q_k(g)\right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{h\in N(g')} \left[Q_h(g) + Q_h(g+g_{ij})\right]\frac{\gamma n''(N-2n'+1)}{N+1} \\
&\quad + \sum_{l\in N(g'')} \left[Q_l(g) + Q_l(g+g_{ij})\right]\frac{\gamma n'(N-2n''+1)}{N+1} \\
&\quad + \sum_{k\notin N(g')\cup N(g'')} \left[Q_k(g) + Q_k(g+g_{ij})\right]\frac{(-2\gamma n'n'')}{N+1}
\end{aligned}$$

$$\geq \sum_{h\in N(g')} \left[Q_h(g) + Q_h(g+g_{ij})\right]\frac{\gamma n''(N-2n'+1)}{N+1}$$

$$
\begin{aligned}
&+ \sum_{h \in N(g'')} [Q_h(g) + Q_h(g + g_{ij})] \frac{\gamma n'(N - 2n'' + 1)}{N+1} \\
&- \sum_{k \notin N(g') \cup N(g'')} [Q_k(g) + Q_k(g + g_{ij})] \frac{2\gamma n' n''}{N+1}
\end{aligned}
$$

$$
\begin{aligned}
\geq & \sum_{h \in N(g')} [Q_h(g) + Q_h(g + g_{ij})] \frac{\gamma n''(N - 2n' + 1)}{N+1} \\
&+ \sum_{h \in N(g'')} [Q_h(g) + Q_h(g + g_{ij})] \frac{\gamma n'(N - 2n'' + 1)}{N+1} \\
&- \sum_{h \notin N(g') \cup N(g'')} [Q_h(g) + Q_h(g + g_{ij})] \frac{2\gamma n' n''}{N+1}
\end{aligned}
$$

$$
\begin{aligned}
&= n' [Q_h(g) + Q_h(g + g_{ij})] \frac{\gamma n''(N - 2n' + 1)}{N+1} \\
&\quad + n'' [Q_h(g) + Q_h(g + g_{ij})] \frac{\gamma n'(N - 2n'' + 1)}{N+1} \\
&\quad - (N - \quad {}' \cancel{n}\, n'') [Q_h(g) + Q_h(g + g_{ij})] \frac{2\gamma n' n''}{N+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{[Q_h(g) + Q_h(g + g_{ij})]}{N+1} \times \gamma n'' n' ((N - 2n' + 1) \\
&\quad + (N - 2n'' + 1) - 2(N - n' - n''))
\end{aligned}
$$

$$
= \frac{2\gamma n' n''}{N+1} [Q_h(g) + Q_h(g + g_{ij})] > 0
$$

This proves the proposition. △

8 References

[1]. V. Bala and S. Goyal (1998), A Non-cooperative theory of network formation, Econometric Institute Report 9855/A. Revised version of 'Self-organization in communication networks', (1997).

[2]. F. Bloch (1995),Endogenous structures of association in oligopolies, *Rand Journal of Economics*, 26, 537-556.

[3]. F. Bloch (1996), Sequential formation of coalitions in games with externalities and fixed payoff division payoffs, *Games and Economic Behavior*, 14, 90-123.

[4]. S. Boorman (1975), 'A combinatorial optimization model for transmission of job information through contact networks', *Bell Journal of Economics*, 6, 1, 216-49.

[5]. S. Goyal (1993), Sustainable communication networks, Tinbergen Institute Discussion Paper, Rotterdam, TI 93-250.

[6]. M. Jackson and A. Wolinsky (1996), A strategic model of social and economic networks, *Journal of Economic Theory*, 71, 1, 44-74.

[7]. R. Krannert and D. Minehart (1998), Buyer-Seller Networks, mimeo Boston University, Dept of Economics.

[8]. R. Myerson [1977] Graphs and cooperation in games, *Mathematics of Operations Research*, 2, 225-229.

[9]. D. Ray and R. Vohra (1997), Equilibrium binding agreements, *Journal of Economic Theory*, 73, 30-78.

[10]. A. Watts (1997), A Dynamic Model of Network Formation, mimeo Vanderbilt University, Department of Economics

[11]. S. Yi (1996), Endogenous Formation of customs unions under imperfect competition: open regulation is good, *Journal of International Economics*, 41, 153-177.

[12]. S. Yi and H. Shin (1995) Endogenous Formation of coalitions in oligopoly, mimeo Dartmouth College, Department of Economics.

[13]. Yi (1997), Stable Coalition Structures with Externalities, *Games and Economic Behavior*, 20, 201-237.

CENTRE FOR DEVELOPMENT ECONOMICS
WORKING PAPER SERIES

No.	Author(s)	Title
1	Kaushik Basu Arghya Ghosh Tridip Ray	The Babu and The Boxwallah : Managerial Incentives and Government Intervention (January 1994). Review of Development Economics, 1997
2	M.N. Murty Ranjan Ray	Optimal Taxation and Resource Transfers in a Federal Nation (February 1994)
3	V. Bhaskar Mushtaq Khan	Privatization and Employment : A Study of The Jute Industry in Bangladesh (March 1994). American Economic Review, March 1995, pp. 267-273
4	V. Bhaskar	Distributive Justice and The Control of Global Warming (March 1994) The North, the South and the Environment: V. Bhaskar and Andrew Glyn (Ed.) Earthscan Publication London, February 1995
5	Bishnupriya Gupta	The Great Depression and Brazil's Capital Goods Sector: A Re-examination (April 1994). Revista Brasileria de Economia 1997
6	Kaushik Basu	Where There Is No Economist: Some Institutional and Legal Prerequisites of Economic Reform in India (May 1994)
7	Partha Sen	An Example of Welfare Reducing Tariff Under Monopolistic Competition (May 1994), Reveiw of International Economics, (forthcoming)
8	Partha Sen	Environmental Policies and North-South Trade : A Selected Survey of the Issues (May 1994)
9	Partha Sen Arghya Ghosh Abheek Barman	The Possibility of Welfare Gains with Capital Inflows in A Small Tariff-Ridden Economy (June 1994)
10	V. Bhaskar	Sustaining Inter-Generational Altruism when Social Memory is Bounded (June 1994)
11	V. Bhaskar	Repeated Games with Almost Perfect Monitoring by Privately Observed Signals (June 1994)

No.	Author(s)	Title
39	Partha Sen	Asset Bubbles in a Monopolistic Competitive Macro Model (June 1996)
40	Pami Dua Stephen M. Miller David J. Smyth	Using Leading Indicators to Forecast US Home Sales in a Bayesian VAR Framework (October 1996)
41	Pami Dua David J. Smyth	The Determinants of Consumers' Perceptions of Buying Conditions for Houses (November 1996)
42	Aditya Bhattacharjea	Optimal Taxation of a Foreign Monopolist with Unknown Costs (January 1997)
43	M. Datta-Chaudhuri	Legacies of the Independence Movement to the Political Economy of Independent India (April 1997)
44	Suresh D. Tendulkar T. A. Bhavani	Policy on Modern Small Scale Industries: A Case of Government Failure (May 1997)
45	Partha Sen	Terms of Trade and Welfare for a Developing Economy with an Imperfectly Competitive Sector (May 1997)
46	Partha Sen	Tariffs and Welfare in an Imperfectly Competitive Overlapping Generations Model (June 1997)
47	Pami Dua Roy Batchelor	Consumer Confidence and the Probability of Recession: A Markov Switching Model (July 1997)
48	V. Pandit B. Mukherji	Prices, Profits and Resource Mobilisation in a Capacity Constrained Mixed Economy (August 1997)
49	Ashwini Deshpande	Loan Pushing and Triadic Relations (September 1997)
50	Rinki Sarkar	Depicting the Cost Structure of an Urban Bus Transit Firm (September 1997)
51	Sudhir A. Shah	Existence and Optimality of Mediation Schemes for Games with Communication (November 1997)
52	V. Pandit	A Note on Data Relating to Prices in India (November 1997)
53	Rinki Sarkar	Cost Function Analysis for Transportation Modes: A Survey of Selective Literature (December 1997)

No.	Author(s)	Title
54	Rinki Sarkar	Economic Characteristics of the Urban Bus Transit Industry: A Comparative Analysis of Three Regulated Metropolitan Bus Corporations in India (February 1998)
55	Aditya Bhattacharjea	Was Alexander Hamilton Right? Limit-pricing Foreign Monopoly and Infant-industry Protection (February 1998)
56	Bishwanath Goldar Badal Mukherji	Pollution Abatement Cost Function: Methodological and Estimation Issues (March 1998)
57	Smita Misra	Economies of Scale in Water Pollution Abatement: A Case of Small-Scale Factories in an Industrial Estate in India (April 1998)
58	T.A. Bhavani Suresh D. Tendulkar	Determinants of Firm-level Export Performance: A Case Study of Indian Textile Garments and Apparel Industry (May 1998)
59	Partha Sen	Non-Uniqueness In The First Generation Balance of Payments Crisis Models (December 1998)
60	Ranjan Ray J.V. Meenakshi	State-Level Food Demand in India: Some Evidence on Rank-Three Demand Systems (December 1998)
61	Brinda Viswanathan	Structural Breaks in Consumption Patterns: India, 1952-1991 (December 1998)
62	Pami Dua Aneesa I. Rashid	Foreign Direct Investment and Economic Activity in India (March 1999)
63	Pami Dua Tapas Mishra	Presence of Persistence in Industrial Production : The Case of India (April 1999)
64	Dr. Sumit Joshi Dr. Sanjeev Goyal	Collaboration and Competition in Networks (May, 1999)