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The Stochastic Turnpike Property without Uniformity in Convex Aggregate Growth Models

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ABSTRACT

An important stochastic turnpike property in optimal growth models asserts that optimal programs of capital accumulation from different initial stocks converge almost surely in a suitable metric. Its proof requires constructing a value-loss process satisfying both uniform boundedness in expectation and sensitivity (in the sense of recording a strictly positive value-loss when the capital stocks being compared diverge). Uniformity assumptions strengthen sensitivity by ensuring that value-loss is independent of time and state of environment in which the divergence occurs. They are imposed either directly on the value-loss process, or indirectly through bounds on the degree of concavity of the felicity or

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production functions, and are acknowledged as strong restrictions on the model. This paper argues, within the context of a convex aggregate growth model, that uncertainty can obviate the need for uniformity. The multiplicity of states afforded by a stochastic framework permits constructing a value-loss process over an "extended" time-line that is a martingale and, hence, relatively easy to uniformly bound in expectation. Further, if capital stocks diverge by some critical amount in any time and state, then the martingale registers an upcrossing across a band of uniform width on its extended time-line for that state thereby giving uniform value-loss. Probabilistic arguments based on the Martingale Upcrossing theorem and the Borel-Cantelli lemma then clinch the turnpike property.

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1 Introduction

Optimal growth theory, in Ramsey-type normative models with convex preferences and technology, has identified an important stability property referred to as the *late* turnpike. It asserts that two infinite horizon optimal programs of capital accumulation from distinct initial stocks converge (almost surely) in a suitable metric. A critical input in the derivation of this property is a strong *uniformity assumption*(Brock and Majumdar [8], Brock and Scheinkman [11], Chang [12], Föllmer and Majumdar[14], Joshi [17], Majumdar and Zilcha [20], McKenzie [22])[1]. It is generally invoked in both discounted and undiscounted frameworks which eschew the time-stationarity restriction on preferences, technology and the evolution of the stochastic environment. It engenders independence from both time and state (of the stochastic environment) of the sensitivity of a key process – the value-loss process – which tracks the divergence in optimal programs from different initial stocks. The primary objective of this paper is to demonstrate, within the context of a convex aggregate *stochastic* growth model, that the late turnpike property can be derived without the encumbrance of a uniformity restriction.

A general derivation of the late turnpike property, without any concession to uniformity, is warranted by the preeminent position this result occupies in the various strands of the growth literature. To substantiate, we offer a brief review.

(i) *Optimal Growth Theory*: The late turnpike property is central by virtue of asserting the global asymptotic stability of optimal programs.[2] Under the assumption of time-stationary preferences, technology and the stochastic environment, and with the added restriction of no discounting, it has been shown in Brock and Mirman [9], Dana [13] and Mirman and Zilcha [24] that all *good* programs converge in an appropriate topology to the golden rule (or optimal stationary) program[3]. In the discounted context, as exemplified in Brock and Mirman [10] and Mirman and Zilcha [25], time-stationarity yields convergence in distribution of optimal programs from distinct initial stocks to the modified golden rule program. In non-stationary models, the uniformity restriction has been critical in establishing convergence in probability (Brock and Majumdar [8]) or the stronger property of almost sure convergence

[1]Optimal growth theory has identified two other turnpike properties – the *early* and the *middle* – of which the latter also relies on a uniformity assumption (for instance, McKenzie [22]).

[2]A comprehensive review of the deterministic literature on turnpike theory, along with a discussion of the role of uniformity, is provided in McKenzie [23]. The review, however, does not cover the stochastic case. For this reason, we have mostly limited the discussion here to optimal growth under uncertainty.

[3]Good programs, first identified in Gale [15], are feasible programs which do not perform infinitely worse in utility terms than the golden rule program and which include non-stationary programs that are optimal in terms of catching-up or overtaking of partial utility sums.

(Chang [12], Föllmer and Majumdar [14], Joshi [17] and Majumdar and Zilcha [20]).

(ii) *Competitive Equilibrium and the Turnpike Property*: In contrast to normative models of optimal growth, the deterministic analysis of Bewley [4], and its stochastic extension by Marimon [21], consider a positive model of equilibrium growth with finitely many (infinitely-lived) consumers and perfectly competitive firms. A competitive equilibrium in their model corresponds to the solution of an optimal growth problem where the social welfare function is a weighted sum of the consumers' utility functions, the weights being the inverses of the marginal utility of expenditures in equilibrium. In particular, the *stationary* competitive equilibrium with transfer payments corresponds to the (modified) golden rule program with respect to this social welfare function. The late turnpike property highlights the global asymptotic stability of *interior* competitive equilibria by showing that they converge to the stationary competitive equilibrium with transfer payments for sufficiently high values of the discount factor.

(iii) *Imperfectly Competitive Equilibria of Endogenous Growth Theory*: The voluminous literature on endogenous growth, following the seminal contributions of Lucas [18] and Romer [31], has considered the dynamic general equilibria of imperfectly competitive markets characterized by sustained growth at endogenously determined levels. The late turnpike property addresses the issue of whether the time path of an imperfectly competitive equilibrium converges to the path of balanced growth. Another facet of this literature has been to explain the difference in growth rates of developing and developed economies (Barro and Sala-i-Martin [2]). Identifying conditions under which long run convergence to the same growth rate does (or does not) obtain bears formal similarity to the late turnpike property.

(iv) *Patience and Chaos*: The late turnpike property rules out the possibility of optimal programs exhibiting chaotic dynamics. In reduced form models with two or more sectors, the existence of complicated dynamics has generally been obtained for low values of the discount factor (Boldrin and Montrucchio [6], Mitra [26], Nishimura, Sorger and Yano [30] and Sorger [32]). Some additional features include felicity functions that are concave but not *strictly* concave, and optimal programs that are possibly non-interior. In an aggregate growth model, Majumdar and Mitra [19] have shown the existence of complicated dynamics when the felicity function depends on both consumption and the capital stock. This raises the following issue: in *aggregate* growth models with strictly concave felicity functions that depend on consumption alone, and in which optimal programs are interior, does the late turnpike property obtain for all values of the discount factor? Majumdar and Zilcha [20] answered this in the affirmative under a uniform lower bound on the degree of concavity of the production function and a particular relative distance function to measure the divergence in capital stocks. This paper attempts to extend the result to the case with no uniformity restrictions and no restriction to a particular metric.

(v) *Applications of Turnpike Theory*: Long run invariance properties, akin to the late turnpike property, exist in diverse areas of economics. For instance, in public finance, a classical invariance proposition states that in a neoclassical growth model a capital income tax is completely shifted to the labour input in the long run (Becker [3]). In the turnpike vernacular, the time paths of the after-tax return to capital per unit, corresponding to different rates of capital income taxation, converge almost surely. Similarly, in industrial organization, a result bearing formal resemblance to the late turnpike property, identifies conditions under which two firms with different initial conditions (technology gap) can close the gap over time through optimal investment in research and development (Budd, Harris and Vickers [7]). The turnpike technique can be profitably applied to all such areas where the objective is to obtain the asymptotic stability of the time paths of a variable of interest generated from different initial conditions.

The pervasive nature of the late turnpike property in economic dynamics, as attested by the above review, provides a compelling reason to re-examine this issue under the greatest generality. In this regard, uniformity assumptions pose a strong restriction on optimal growth models. This paper demonstrates that the potential to exploit multiple states of the environment afforded by a stochastic paradigm can eradicate the need for any strong uniformity restriction. Towards this end, we organize the paper as follows. The convex optimal growth model is presented in Section 2. The nature of the uniformity assumption is examined in Section 3. A non-technical description of our methodology is provided in Section 4. The mathematical underpinnings of this methodology, along with a formal statement of results, is available in Section 5. All proofs are relegated to an Appendix. Our conclusions are contained in Section 6.

2 The Aggregate Growth Model

Our description of the growth model generalizes Brock and Mirman [10], Majumdar and Zilcha [20], and Mirman and Zilcha [24] by allowing non-stationarities in preferences, technology and the evolution of the stochastic environment. From now on, we will let $\mathcal{I}_+ = \{0, 1, 2, \ldots\}$, and let $\langle h_t \rangle$ denote the sequence, $h_0, h_1, h_2, \ldots, h_t, \ldots,\ t \in \mathcal{I}_+$.

The possible states, ω_t, of the environment at any date $t \in \mathcal{I}_+$ is given by an uncountably infinite set, Ω_t, that is a compact metric space in an appropriate topology. Let $\mathcal{E}_t$ denote the Borel σ-field of subsets of Ω_t generated by the open sets in this topology. By assumption, Ω_t

satisfies the second countability axiom (Munkres [29, Section 4-1]), i.e. Ω_t has a countable basis, $\mathcal{H}_t = \langle H_{n,t} \rangle$, $n \in \mathcal{I}_+$, for its topology. It follows that $\mathcal{E}_t = \sigma(\mathcal{H}_t)$. Let $\Omega = \times_{t=0}^{\infty} \Omega_t$ denote the set of all sequences, $\omega = \langle \omega_t \rangle$, $\omega_t \in \Omega_t$, and $\mathcal{F}$ denote the σ-field on Ω generated by open sets in the product topology on Ω. The *stochastic environment* is represented by the probability space $(\Omega, \mathcal{F}, \nu)$, where ν is a probability measure on Ω. Let $\langle \mathcal{F}_t \rangle$ denote the filtration on Ω. $\mathcal{F}_t$ is the sub-σ-field on Ω induced by partial history till date t, i.e. $\mathcal{F}_t = \sigma(\mathcal{E}_0 \times \mathcal{E}_1 \times \cdots \times \mathcal{E}_t \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots)$.[4]

Technology is described by a sequence of possibly time-varying production functions $\langle f_t \rangle$, $f_t : \mathcal{R}_+ \times \Omega_{t+1} \to \mathcal{R}_+$, where for each $t \in \mathcal{I}_+$:

(A.1) f_t is continuous on $\mathcal{R}_+ \times \Omega_{t+1}$.

(A.2) For each $\omega_{t+1} \in \Omega_{t+1}$, $f_t(0, \omega_{t+1}) = 0$, $f_t(k, \omega_{t+1})$ is strictly concave for $k \geq 0$, and $f_t'(k, \omega_{t+1}) \equiv \partial f_t(k, \omega_{t+1})/\partial k$ exists and is strictly positive for $k > 0$.

Preferences are represented by a sequence of possibly time-varying felicity functions $\langle u_t \rangle$, $u_t : \mathcal{R}_+ \to \mathcal{R}$, such that for each $t \in \mathcal{I}_+$:

(A.3) $u_t(c)$ is continuous and strictly concave for $c \geq 0$.

(A.4) $u_t'(c)$ exists and is strictly positive for $c > 0$ with $u_t'(c) \uparrow +\infty$ as $c \downarrow 0$.

The initial stock, s, is random and is drawn from the set $\mathcal{L} \equiv \mathcal{L}_\infty(\Omega, \mathcal{F}_0, \nu, \mathcal{R}_{++})$ of all essentially bounded $\mathcal{F}_0$-measurable functions from Ω into $\mathcal{R}_{++}$. A real-valued $\langle \mathcal{F}_t \rangle$-adapted process, $\langle (k_t, c_t) \rangle$, is a *feasible program* from $s \in \mathcal{L}$ if with probability 1:

$$k_0 + c_0 \leq s \tag{1}$$

$$k_{t+1} + c_{t+1} \leq f_t(k_t, \omega_{t+1}), \quad t \in \mathcal{I}_+ \tag{2}$$

$$k_t \geq 0, \quad c_t \geq 0, \quad t \in \mathcal{I}_+ \tag{3}$$

[4] Note that $\mathcal{E}_0 \times \mathcal{E}_1 \times \cdots \times \mathcal{E}_t \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots$ denotes the collection of cylindrical sets of the form, $A_0 \times A_1 \times \cdots \times A_t \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots$, where $A_i \in \mathcal{E}_i$ for $i = 0, 1, \ldots, t$.

The set of all feasible programs from a given initial stock $s \in \mathcal{L}$ is denoted by $\Phi(s)$. A program, $\langle (k_t^s, c_t^s) \rangle \in \Phi(s)$, is *optimal* if for any other program, $\langle (k_t, c_t) \rangle \in \Phi(s)$:

$$\limsup_{N \to \infty} \sum_{t=0}^{N} E\left[u_t(c_t) - u_t(c_t^s)\right] \leq 0$$

Existence of an optimal program follows under a joint boundedness restriction on preferences and technology (Majumdar and Zilcha [20, Theorem 1] or Mitra and Nyarko [27, Condition E]). From assumption (A.4), optimal programs are interior and (in addition to appropriate transversality conditions for discounted and undiscounted models) satisfy the stochastic Euler equations:

$$u_t'(c_t^s) = E\left[u_{t+1}'(c_{t+1}^s) f_t'(k_t^s, \omega_{t+1}) \mid\mid \mathcal{F}_t\right] \quad \nu - a.s. \; t \in \mathcal{I}_+ \tag{4}$$

Define the competitive price process, $\langle p_t^s \rangle$, associated with the optimal program, $\langle (k_t^s, c_t^s) \rangle$, as $p_t^s = u_t'(c_t^s)$, $t \in \mathcal{I}_+$. Further, let:

$$\pi_0^s = 1, \quad \pi_{t+1}^s = \prod_{i=0}^{t} f_i'(k_i^s, \omega_{i+1}), \qquad t \in \mathcal{I}_+$$

where π_t^s is $\mathcal{F}_t$-measurable and strictly positive (almost surely) from the interiority of optimal programs. Multiplying both sides of (4) by π_t^s, and using its $\mathcal{F}_t$-measurability, the Euler equations can be rewritten succinctly as:

$$p_t^s \pi_t^s = E\left[p_{t+1}^s \pi_{t+1}^s \mid\mid \mathcal{F}_t\right] \quad \nu - a.s. \; t \in \mathcal{I}_+ \tag{5}$$

Therefore, the process $\langle p_t^s \pi_t^s \rangle$ is a $\langle \mathcal{F}_t \rangle$-martingale with $E[p_t^s \pi_t^s] = E[p_0^s \pi_0^s] = E[u_0'(c_0^s)]$, $t \in \mathcal{I}_+$. We can interpret $\langle p_t^s \pi_t^s \rangle$ as the sequence of valuations of future increments of a unit of capital invested optimally in time period 0 from the initial stock s.

Consider an initial stock $y \in \mathcal{L}$, $y \neq s$, and let $\mu \equiv E[u_0'(c_0^s) - u_0'(c_0^y)]$. An examination of (5) shows that adding a constant to both sides of the Euler equation leaves it unaffected. However, given that the two sides of (5) are strictly positive (almost surely), this operation conveniently bounds each side from below by the constant. Letting $M_t^i \equiv p_t^i \pi_t^i + \mu$, $i = s, y$, the Euler equations become $M_t^i = E[M_{t+1}^i \mid\mid \mathcal{F}_t]$ almost surely for each $t \in \mathcal{I}_+$.

3 The Uniformity Assumption

In this section, to illuminate the precise nature of the uniformity assumption, we review the methodology that underlies the stochastic late turnpike property in non-stationary models.

The first step entails the construction of a value-loss process, $\langle V_t \rangle$, by utilizing the competitive conditions which characterize an optimal program (the "turnpike"). These are the Euler conditions in aggregate models (Joshi [17], Majumdar and Zilcha [20]) and the reduced utility maximization conditions in multisector models (Brock and Majumdar [8], Chang [12], Föllmer and Majumdar [14]). The process $\langle V_t \rangle$ has the convenient property of being either a martingale (Majumdar and Zilcha [20, Equations 6.20 and 6.21]), a submartingale (Brock and Majumdar [8]) or a supermartingale (Joshi [17], Marimon [21]) thereby permitting a passage to the rich theory of martingales. The second step is to ensure that $\langle V_t \rangle$ is uniformly bounded in expectation. This is achieved in aggregate models through interiority of optimal programs, and in multisector models through the transversality (bounded capital value) condition. The third step is to endow $\langle V_t \rangle$ with the sensitivity to record a strictly positive difference – called *value-loss* – when optimal programs from different initial stocks diverge by some pre-specified critical amount. This value-loss will in general depend on the time period, t, and the state of environment, ω, in which the divergence occurs.

At this stage, as noted by Föllmer and Majumdar [14, Theorem 3.1], a weak version of the late turnpike property can be obtained: for any arbitrary constant $\lambda > 0$, $\langle V_t \rangle$ will almost surely leave a set on which value-loss exceeds λ in finite time. This is a consequence of the uniform bound on the expectation of $\langle V_t \rangle$ and the Martingale Convergence theorem (Billingsley [5, Theorem 35.4]). Coupled with the sensitivity property, it implies that capital stocks cannot diverge for infinitely many periods by an amount that causes value-loss to exceed λ. It is a weak characterization, however, because convergence is not implied: capital stocks can diverge for infinitely many periods by any amount that causes value-loss to be less than λ.

This is precisely the point where the uniformity assumption enters into the analysis to force the convergence of optimal programs from different initial stocks by strengthening the sensitivity of $\langle V_t \rangle$. In particular, for any $\epsilon > 0$, if capital stocks diverge by more than ϵ in period t and state of environment ω, then uniformity dictates that $\langle V_t \rangle$ record a value-loss of at least $\eta(\epsilon) > 0$, where $\eta(\epsilon)$ is independent of the tuple (t, ω). Since $\langle V_t \rangle$ converges almost surely from the Martingale Convergence theorem, the (contrapositive of the) uniformity assumption ensures that capital stocks generated by optimal programs from different initial stocks converge too. This is how the twin properties of uniformly bounded expectation (which allows an application of the Martingale Convergence theorem to $\langle V_t \rangle$) and the uniformity assumption (which ties the convergence of optimal programs from distinct initial stocks to convergence of $\langle V_t \rangle$) act in conjunction to yield the late turnpike property.

A potential conflict exists, however, between the dictates of uniform boundedness on the one hand and uniform sensitivity on the other. This tension exists because it is in general difficult to reconcile a process that is uniformly bounded in expectation with one that can *a priori* register an infinite number of jumps of a magnitude exceeding some strictly positive constant. The construction of a value-loss process, therefore, while imparting a martingale structure to $\langle V_t \rangle$, has to balance these vital but conflicting objectives of uniform boundedness and uniform sensitivity.

While the uniform bound on expectation is relatively easier to impose, ensuring uniform sensitivity poses the difficult problem of identifying the precise restrictions on the fundamentals of the model – the felicity functions, the production technology, and the discount factor – that permit value-loss to record a uniform jump in those states of the environment where capital stocks diverge while being uniformly bounded on average across all states. The literature has addressed the problem in two ways which may be classified as *direct* and *indirect*. The former method directly imposes uniform sensitivity on the value-loss process by appealing to appropriate curvature restrictions on technology and preferences without making them explicit (Brock and Majumdar [8], Chang [12], Föllmer and Majumdar [14], Joshi [17], McKenzie [22]). The latter method proves uniform sensitivity from first principles by explicitly imposing bounds on the degree of concavity of the felicity functions (Brock and Scheinkman [11], Guerrero-Luchtenberg [16] and McKenzie [22] in the multisector case) or the production functions (Majumdar and Zilcha [20] in the aggregate case).

In either their direct or indirect guise, uniformity assumptions are acknowledged as constituting strong restrictions on the growth model. In the direct approach, they have been characterized as "strong uniformity" (Brock and Majumdar [8, Assumption (A.4)]) or as a "strong value-loss assumption" (Föllmer and Majumdar [14, p.281]). Further, as noted in Mckenzie [22], they are difficult to extend to discounted models without additional restrictions on the discount factor. In the indirect approach, they preclude growth models with time-varying preferences and technology that asymptotically approach the linear case. For instance, consider the sequence of functions $\langle h_t : \mathcal{R}_+ \rightarrow \mathcal{R}_+ \rangle$, $h_t(x) = x^{1-1/(t+2)}$. This sequence of strictly increasing, strictly concave functions is precluded from describing felicity or production functions. The degree of concavity of h_t, given by $-xh_t''(x)/h_t'(x)$, is equal to $1/(t+2)$, and approaches zero as $t \rightarrow \infty$. The uniformity assumption, however, requires that the degree of concavity of each h_t be uniformly bounded from below by a strictly positive constant.

4 Methodology: A Descriptive View

In this section, we offer an informal description of our method. Recall that our fundamental problem lies in balancing a uniform bound on the expectation of $\langle V_t \rangle$ with its potential to possibly execute an infinite number of jumps of uniform size in those time periods and states where capital stocks diverge by some critical amount. This can be visualized as the problem of attempting to uniformly bound (in expectation) an increasing step-function on an unbounded time interval. The crux of our approach is relatively simple: while it is difficult to *uniformly* bound a strictly increasing (decreasing) process, it is fairly easy to uniformly bound a process in which every up-jump (down-jump) is immediately followed by an equal-sized down-jump (up-jump); this negation of the initial jump ensures that the process starts at the same level once again and any pre-specified uniform upper (lower) bound is not compromised even though the process is permitted an infinite number of jumps.

Figure 1 Somewhere Here

To see how we apply this idea to $\langle V_t \rangle$, consider Figure 1. The graph on top tracks the time path of $||k_t^s(\omega) - k_t^y(\omega)||$, where $||.||$ is any distance function, permitting us to observe those time periods when the divergence in capital stock exceeds some arbitrary constant $\epsilon > 0$ (in our example, $t = 1, 4, 5$). The graph for $\langle V_t \rangle$ shows that for these time periods of more than ϵ-divergence, value-loss exceeds the strictly positive constant, $2\mu > 0$.[5] In the other time periods, when there is less than ϵ-divergence (in our example, $t = 0, 2, 3$), value-loss is non-positive. We then extend the time-line by including the mid-point between periods t and $t+1$ for $t = 0, 1, 2, \ldots$, and posit a constant value (equal to zero) for value-loss, $V_{t+\frac{1}{2}}$, at this intermediate point. We let $\tau = 2t$, $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots$, denote this extended time-line and let $\langle V_\tau \rangle$ denote the value-loss process over τ. Since we will have occasion to switch between our original and new time-lines, it is useful to note that even-valued τ correspond to the original time-line while odd-valued τ refer to the fictitiously introduced "intermediate" points on the time-line.

We now observe that if we have more than ϵ-divergence in capital stocks, for instance in $t = 1$, then $\langle V_\tau \rangle$ moves upwards across a band of width equal to 2μ over the time interval $t = \frac{1}{2}$ to $t = 1$ (or, in the lexicon of martingale theory, it registers an *upcrossing*[6] over the range $[0, 2\mu]$ during this time interval). This upcrossing, by construction, is immediately

[5] We will assume in the sequel that $s \leq y$ almost surely, with $s < y$ on some set of strictly positive measure. This will ensure that $\mu \equiv E[u_0'(c_0^s) - u_0'(c_0^y)] > 0$.

[6] In martingale theory, an upcrossing across an interval can take place over more than one time period while in our analysis it takes place in one time period. We use the terminology of an upcrossing because

followed by a *downcrossing* of equal size from $t = 1$ to $t = 1\frac{1}{2}$ so that value-loss returns to its initial level once again. If there is less than ϵ-divergence in any period, for instance in $t = 2$, then V_τ remains non-positive and no upcrossing is registered across $[0, 2\mu]$; from $t = 2$ to $t = 2\frac{1}{2}$ value-loss returns to its base-line level of zero.

In implementing the above idea, we have to ensure that $\langle V_\tau \rangle$ is a martingale. The multiplicity of states afforded by a stochastic paradigm permits us the facility to induce upcrossings across $[0, 2\mu]$ of $\langle V_\tau \rangle$ for those realizations where $||k_t^s(\omega) - k_t^y(\omega)|| > \epsilon$ while simultaneously manipulating those states where $||k_t^s(\omega) - k_t^y(\omega)|| \leq \epsilon$ in order to endow $\langle V_\tau \rangle$ with a martingale structure.[7] The construction has the flavor of an optimal stopping argument. Suppose $\langle (k_t^y, c_t^y) \rangle$ is the "turnpike", and the planner (or representative consumer) is off the turnpike on the path $\langle (k_t^s, c_t^s) \rangle$. Staying off the turnpike subjects the planner to value-loss and, therefore, the objective is to switch from $\langle (k_t^s, c_t^s) \rangle$ to the turnpike. However, such a switch involves an adjustment cost which depends on the distance of $\langle (k_t^s, c_t^s) \rangle$ from the turnpike. Hence, given a realization ω, the planner would like to "stop" value-loss and effect the transition to the turnpike at time $t(\omega)$ if $||k_{t(\omega)}^s(\omega) - k_{t(\omega)}^y(\omega)|| \leq \epsilon$ and not if $||k_{t(\omega)}^s(\omega) - k_{t(\omega)}^y(\omega)|| > \epsilon$. In an optimal stopping problem, the objective is to determine the optimal stopping time to effect the switch to the turnpike, given the adjustment cost. Our construction considers the reverse problem: given any $t \in \mathcal{I}_+$ and realization ω, what is the size of the adjustment cost which just makes it optimal for the planner to "stop" value-loss at time t if $||k_t^s(\omega) - k_t^y(\omega)|| \leq \epsilon$ but not on the complement of this set. The value-loss process is shown to imbibe a martingale structure as a consequence of such a choice of adjustment costs. This argument of course presupposes that in *each* period $t \in \mathcal{I}_+$ there will exist a set of states of strictly positive measure on which $||k_t^s(\omega) - k_t^y(\omega)|| \leq \epsilon$. This technical requirement is met via a reachability assumption on initial stocks.

We have now achieved our twin objectives: the uniform bound on the expectation of $\langle V_\tau \rangle$ is realized because a potentially unbounded increasing process is replaced with an oscillatory martingale process, and martingales have the convenient property that their expectation in any period $\tau > 0$ is equal to their expectation in period $\tau = 0$. Interiority of the optimal programs will ensure that expectation in period $\tau = 0$ is finite. Uniform value-loss is attained because, when capital stocks are more than ϵ-distance apart, $\langle V_\tau \rangle$ upcrosses a band of uniform and strictly positive width.

of the formal analogy between our resolution of the turnpike property and the Martingale Upcrossing theorem (Billingsley [5, Theorem 35.3]).

[7]Note that optimal programs are defined on the original time-line while the value-loss process is constructed over the extended time-line.

5 Methodology: A Formal Analysis

Our construction of a value-loss process has to be tempered by four requirements: $\langle \mathcal{F}_t \rangle$-adaptability, the martingale property, uniform bound on expectation, and upcrossing across a band of uniform width following a critical divergence in capital stocks. For each $t \in \mathcal{I}_+$, we isolate a collection of subsets of Ω generating $\mathcal{F}_t$ and construct a value-loss process satisfying the requisite properties with respect to this collection. We then invoke Dynkin's $\pi - \lambda$ theorem (Billingsley [5, Theorem 3.2]) to extend these properties to the σ-field $\mathcal{F}_t$ generated by this collection. As a first step, we exploit the second countability axiom to explicitly characterize $\mathcal{F}_t$ in terms of the countable basis of Ω_i, $i = 0, 1, \ldots, t$.

Lemma 1 *Let Ω_t, $t \in \mathcal{I}_+$, be a compact metric space and consider the countable collection $\mathcal{B}_t = \mathcal{H}_0 \times \mathcal{H}_1 \times \cdots \times \mathcal{H}_t \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots$, $t \in \mathcal{I}_+$. Then, $\mathcal{F}_t = \sigma(\mathcal{B}_t)$. Further, $\mathcal{B}_t$ can be assumed to be a partition of Ω.*

Consider the set, $S_t = \{\omega : k_t^s(\omega) \neq k_t^y(\omega)\}$, and its subset, $S_t^\epsilon = \{\omega : ||k_t^s(\omega) - k_t^y(\omega)|| > \epsilon\}$, $t \in \mathcal{I}_+$. Both sets are $\mathcal{F}_t$-measurable by virtue of the $\langle \mathcal{F}_t \rangle$-adaptability of optimal programs. We would like to ensure that $\nu(\Omega \backslash S_t^\epsilon) > 0$ for each $t \in \mathcal{I}_+$ so that we are assured a set of states of strictly positive measure on which to manipulate the value-loss process. This necessitates a "reachability" restriction on initial stocks. Clearly, in non-stationary models, initial stocks must satisfy some reachability condition if the late turnpike property is to obtain.[8] Our definition of reachability is a modification of the expansibility notion that we can reach one initial stock from another in finite time for some realizations in Ω. Since what transpires over a *finite* horizon may be ignored in analyzing the *asymptotic* behaviour of optimal programs, in effect we require that initial stocks coincide for these realizations.

Definition 1 *Given $s, y \in \mathcal{L}$, $s \leq y$ ν-a.s., consider the $\mathcal{F}_0$-measurable set, $\Delta(s, y) = \{\omega : s(\omega) = y(\omega)\}$. The initial stock y is reachable from s if $\nu(\Delta(s, y)) > 0$.*

Note that reachability, while requiring that the initial stocks agree on some set of strictly positive measure, however small, places no restriction on how much they diverge on the complement of that set. That is, apart from strict positivity, the ν-measure of $\Delta(s, y)$ is not

[8]The precise function of reachability in the proof of the late turnpike property is to bound the value-loss process (McKenzie [22]). Reachability is put towards the same end in our framework, albeit somewhat tangentially. By facilitating the construction of a value-loss process that is a martingale, it indirectly bounds it in expectation.

limited in any manner. Of course, to avoid trivialities, $\nu(\Delta(s,y)) < 1$. In this sense, the reachability restriction is not unduly stringent. The onerous task remains to demonstrate that, starting from initial stocks that coincide on a set of positive but less than full probability measure, optimal programs will be arbitrarily close for all sufficiently large $t \in \mathcal{I}_+$ on a set of *full* probability measure.

Before proceeding, we record without proof a standard monotonicity property of optimal programs (Majumdar and Zilcha [20]). Note that while monotonicity of optimal *capital* input programs also obtains in non-convex growth models (Mitra and Nyarko [27]), monotonicity of optimal *consumption* programs obtains only in the convex growth model.

Lemma 2 *Consider any* $s, y \in \mathcal{L}$, $s \leq y$ ν*-a.s. Under Assumptions* (A.1)-(A.4), $k_t^s \leq k_t^y$ *and* $c_t^s \leq c_t^y$ *almost surely for each* $t \in \mathcal{I}_+$.

We now show that reachability and Lemma 2 implies that $\nu(\Omega \backslash S_t) > 0$ for each $t \in \mathcal{I}_+$. Since $\Omega \backslash S_t \subseteq \Omega \backslash S_t^\epsilon$, it will then follow that $\nu(\Omega \backslash S_t^\epsilon) > 0$ for each $t \in \mathcal{I}_+$.

Lemma 3 *Consider initial stocks* $s, y \in \mathcal{L}$, $s \leq y$ ν*-a.s., such that* y *is reachable from* s. *Under Assumptions* (A.1)-(A.4), $\Delta(s,y) \subseteq \Omega \backslash S_0 \subseteq \ldots \subseteq \Omega \backslash S_t \subseteq \Omega \backslash S_{t+1}$.

Unfortunately, it is not enough to just show that $\nu(\Omega \backslash S_t^\epsilon) > 0$, $t \in \mathcal{I}_+$. What we require is that $\nu((\Omega \backslash S_{t+1}^\epsilon) \cap B_{n,t}) > 0$ for each $B_{n,t} \in \mathcal{B}_t$. We refer to this property as the *non-empty intersection* property.[9] It will permit us to let the value-loss process register a jump of 2μ for realizations in $S_{t+1}^\epsilon \cap B_{n,t}$ while simultaneously manipulating realizations in $(\Omega \backslash S_{t+1}^\epsilon) \cap B_{n,t}$ (through a suitable choice of adjustment costs) such that the martingale property holds. That is, there will be no conflict in reconciling the uniform sensitivity property with the martingale property (and hence uniformly bounded expectation) because each will be addressed over a set of strictly positive measure that is disjoint from the other.

There is no reason, however, why the collection $\mathcal{B}_t$ will exhibit the non-empty intersection property. In general, it will be made up of the following two non-empty sub-collections of mutually disjoint sets:

$$\mathcal{B}_t' = \{B_{n,t}' \in \mathcal{B}_t \,:\, \nu((\Omega \backslash S_{t+1}^\epsilon) \cap B_{n,t}) > 0\} \qquad (6)$$
$$\mathcal{B}_t'' = \{B_{n,t}'' \in \mathcal{B}_t \,:\, \nu((\Omega \backslash S_{t+1}^\epsilon) \cap B_{n,t}) = 0\} \qquad (7).$$

[9] This terminology is for brevity only for in fact the property requires not only non-empty intersection of $\Omega \backslash S_{t+1}^\epsilon$ with each $B_{n,t} \in \mathcal{B}_t$ but also that this intersection have strictly positive measure.

The following lemma combines the above sub-collections in order to derive a countable collection satisfying the non-empty intersection property.

Lemma 4 *For each $t \in \mathcal{I}_+$, there exists a countable partition, $\mathcal{C}_t$, of Ω displaying the non-empty intersection property. Further, $\sigma(\mathcal{C}_t) \subseteq \mathcal{F}_t$.*

Since $\mathcal{C}_t$ does not generate $\mathcal{F}_t$, we need to augment this collection while preserving the non-empty intersection property. For this purpose, re-index sets in $\mathcal{B}''_t$ such that $C_{n,t} \cap B''_{n,t} = \emptyset$ for each $n \in \mathcal{I}_+$. We can now prove:

Lemma 5 *Consider $\mathcal{D}_t = \langle D_{n,t} \rangle$, $D_{n,t} = C_{n,t} \cup B''_{n,t}$ for $C_{n,t} \in \mathcal{C}_t$, $B''_{n,t} \in \mathcal{B}''_t$, $n \in \mathcal{I}_+$. Then $\mathcal{D}_t$ displays the non-empty intersection property and $\sigma(\mathcal{D}_t) = \mathcal{F}_t$.*

The collection $\mathcal{D}_t$ is *not* a partition of Ω. Rather, finite intersections of the elements of $\mathcal{D}_t$, coupled with the operations of finite unions and set-theoretic differences, yields $\mathcal{B}_t$. With this we turn to a consideration of the adjustment cost *parameters* which will play a vital role in ensuring that value-loss is a martingale. These parameters in any period $t+1$ are defined over elements of the collection $\mathcal{D}_t$, $t \in \mathcal{I}_+ \backslash \{0\}$, as follows:

$$\zeta_{t+1}(D_{n,t}) = \frac{\mu\nu(D_{n,t}) - \int_{S^{\epsilon}_{t+1} \cap D_{n,t}} M^{s}_{t+1} d\nu}{\int_{(\Omega \backslash S^{\epsilon}_{t+1}) \cap C_{n,t}} \delta_{t+1}\, d\nu}, \qquad D_{n,t} = C_{n,t} \cup B''_{n,t}, \quad n \in \mathcal{I}_+ \tag{8}$$

$$\xi_{t+1}(D_{n,t}) = \frac{-\mu\nu(D_{n,t}) - \int_{S^{\epsilon}_{t+1} \cap D_{n,t}} M^{y}_{t+1} d\nu}{\int_{(\Omega \backslash S^{\epsilon}_{t+1}) \cap C_{n,t}} \delta_{t+1}\, d\nu}, \qquad D_{n,t} = C_{n,t} \cup B''_{n,t}, \quad n \in \mathcal{I}_+ \tag{9}$$

where $\delta_t \equiv ||k^s_t - k^y_t|| + 1 > 0$. Since $\nu((\Omega \backslash S^{\epsilon}_{t+1}) \cap C_{n,t}) > 0$ for all $C_{n,t} \in \mathcal{C}_t$ from Lemma 4, the cost parameters defined by (8) and (9) are well-defined and finite on $\mathcal{C}_t$. Further, they are $\mathcal{F}_{t+1}$-measurable by construction (since $C_{n,t} \in \mathcal{F}_t \subset \mathcal{F}_{t+1}$).

We now use the adjustment cost parameters to define the adjustment cost *functions* over *realizations* in Ω. Let:

$$\tilde{\zeta}_{t+1}(\omega) = \zeta_{t+1}(D_{n,t}), \quad \omega \in C_{n,t}, \quad D_{n,t} = C_{n,t} \cup B''_{n,t}, \quad n \in \mathcal{I}_+ \tag{10}$$

$$\tilde{\xi}_{t+1}(\omega) = \xi_{t+1}(D_{n,t}), \quad \omega \in C_{n,t}, \quad D_{n,t} = C_{n,t} \cup B''_{n,t}, \quad n \in \mathcal{I}_+ \tag{11}$$

Since C_t is a partition of Ω, the adjustment cost functions are well-defined. By construction, they are $\langle \mathcal{F}_t \rangle$-adapted. Further, they assume a constant value over each disjoint set, $C_{n,t} \in C_t$.

We now construct two complementary processes that together make up the value-loss process. Consider first the real-valued stochastic process, $\langle X_t \rangle$, which measures total cost (value-loss plus adjustment costs) incurred when $\langle (k_t^y, c_t^y) \rangle$ is the "turnpike" and the planner is off the turnpike on the path $\langle (k_t^s, c_t^s) \rangle$. In period t, if the event is S_t^c, then the planner stays off the turnpike incurring a value-loss M_t^s. However, if the event is $\Omega \backslash S_t^c$, then the planner switches to the turnpike eradicating any value-loss but incurring an adjustment cost in the transition which depends on the distance between the capital stocks. This motivates the following definition for the $\langle X_t \rangle$ process, where $\chi(C)$ represents the indicator function of a set C:

$$X_t = \chi(S_t^c) M_t^s + \chi(\Omega \backslash S_t^c) \tilde{\zeta}_t \delta_t \,, \quad t \in \mathcal{I}_+ \backslash \{0\} \tag{12}$$

with the initial condition, $X_0 = \mu$. A symmetric argument is used to construct the real-valued stochastic process $\langle Y_t \rangle$ which measures value-loss when $\langle (k_t^s, c_t^s) \rangle$ is the "turnpike" and the planner is off the turnpike on the path $\langle (k_t^y, c_t^y) \rangle$:

$$Y_t = \chi(S_t^c) M_t^y + \chi(\Omega \backslash S_t^c) \tilde{\xi}_t \delta_t \,, \quad t \in \mathcal{I}_+ \backslash \{0\} \tag{13}$$

with the initial condition, $Y_0 = -\mu$. We now extend the definition of $\langle X_t \rangle$ and $\langle Y_t \rangle$ over the "intermediate" periods $t + \frac{1}{2}$, $t \in \mathcal{I}_+$, by letting:

$$X_{t+\frac{1}{2}}(\omega) = \mu \,, \quad Y_{t+\frac{1}{2}}(\omega) = -\mu \,, \quad \omega \in \Omega \tag{14}$$

We further let $\mathcal{D}_{t+\frac{1}{2}} = \mathcal{D}_t$. Then, $\mathcal{F}_{t+\frac{1}{2}} = \mathcal{F}_t$. Now let $\tau = 2t$, $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, \ldots$, denote the "extended" time-line and $\langle X_\tau \rangle$, $\langle Y_\tau \rangle$, the complementary processes over this new time-line. We can prove:

Proposition 1 *Under Assumptions* (A.1)-(A.4), $\langle X_\tau \rangle$ *and* $\langle Y_\tau \rangle$ *are* $\langle \mathcal{F}_\tau \rangle$*-martingale processes on the time-line* τ *which are uniformly bounded in expectation.*

The economic intuition underlying the above result is as follows. Suppose that $\tau + 1$ is even-valued so that it corresponds to the original time-line. If in this time period the planner decides to stay off the turnpike independent of which event obtains, then $E[M_{\tau+1}^s - M_{\tau+1}^y]$ is a measure of the (expected) relative loss of value by not making a transition to the turnpike. Recalling (5), this is just μ. Further, $E[X_{\tau+1}||\mathcal{F}_\tau]$ is the (conditional) total cost of following the strategy of staying off the turnpike if the event $S^\epsilon_{\tau+1}$ occurs and making the transition to the turnpike if $\Omega \backslash S^\epsilon_{\tau+1}$ occurs. The adjustment cost has been chosen such that $E[X_{\tau+1}||\mathcal{F}_\tau] = \mu$ with probability one. Moreover, note by construction that since τ is odd, we have fixed X_τ at μ for all ω. Therefore, $E[X_{\tau+1}||\mathcal{F}_\tau] = X_\tau$ with probability one, which is simply the martingale property.

Culling together the properties of the above two processes, we have the desired value-loss process on the time-line τ.

Proposition 2 *Under Assumptions* (A.1)-(A.4), *there exists a zero-mean $\langle\mathcal{F}_\tau\rangle$-martingale process $\langle V_\tau\rangle$ such that:*

$$V_\tau(\omega) = \begin{cases} 0, & \tau = 0, & & \omega \in \Omega \\ 0, & \tau = 2n+1, & n \in \mathcal{I}_+, & \omega \in \Omega \\ < 0, & \tau = 2n, & n \in \mathcal{I}_+ \backslash \{0\} & \omega \in \Omega \backslash S^\epsilon_\tau \\ > 2\mu, & \tau = 2n, & n \in \mathcal{I}_+ \backslash \{0\}, & \omega \in S^\epsilon_\tau \end{cases}$$

The middle graph in Figure 1 is the pictorial analog of Proposition 2. For even-valued τ corresponding to the original time-line, $\langle V_\tau\rangle$ upcrosses $[0, 2\mu]$ from period $\tau - 1$ to τ on the set S^ϵ_τ. On the other hand, on the set $\Omega \backslash S^\epsilon_\tau$, the process $\langle V_\tau\rangle$ remains below the horizontal axis and does not register any upcrossing across $[0, 2\mu]$ between periods $\tau - 1$ to τ.

The martingale and upcrossing properties of $\langle V_\tau\rangle$, which are fundamental to the resolution of the late turnpike property, are invariant to an affine transformation of value-loss. Consider any transformed process, $\langle V^{\alpha,\beta}_\tau\rangle$, $\alpha, \beta \in \mathcal{R}_{++}$, where $V^{\alpha,\beta}_\tau(\omega) = \beta V_\tau(\omega) + \alpha$ for all $\omega \in \Omega$ and $\tau \in \mathcal{I}_+$. Any $\langle V^{\alpha,\beta}_\tau\rangle$ can assume the role of a value-loss process since it is a martingale (and hence uniformly bounded in expectation) and registers an upcrossing across the interval $[\alpha, 2\mu\beta + \alpha]$ when there is more than ϵ-divergence in capital stocks. A change in α adjusts the upper and lower limits of the upcrossing-band by the same amount. The positivity of β ensures that we *upcross* a band given some critical divergence in capital stocks; if β was negative, a symmetric argument can be constructed where $\langle V^{\alpha,\beta}_\tau\rangle$ *downcrosses* the band $[2\mu\beta + \alpha, \alpha]$ following a critical divergence. Note that the endpoints of the upcrossing interval do not depend upon ϵ.

For any $\alpha, \beta \in \mathcal{R}_{++}$, and $\omega \in \Omega$, define the process $\langle K^{\alpha,\beta}_\tau\rangle$, $K^{\alpha,\beta}_\tau : \Omega \to \{0, 1\}$, $\tau \in \mathcal{I}_+$, as:

$$K^{\alpha,\beta}_\tau(\omega) = \begin{cases} 0, & \tau = 0 \\ 0, & \tau \in \mathcal{I}_+ \backslash 0, \quad V^{\alpha,\beta}_{\tau-1}(\omega) \geq 2\mu\beta + \alpha \\ 1, & \tau \in \mathcal{I}_+ \backslash 0, \quad V^{\alpha,\beta}_{\tau-1}(\omega) < 2\mu\beta + \alpha \end{cases}$$

This process is predictable (i.e. $K_\tau^{\alpha,\beta}$ is $\mathcal{F}_{\tau-1}$-measurable) and it indicates one upcrossing across $[\alpha, 2\mu\beta+\alpha]$ of $\langle V_\tau^{\alpha,\beta}\rangle$ when a chain of 1's is followed by a 0. This is shown in the third graph in Figure 1. Note that in whatever manner the process may oscillate, only *one* upcrossing is registered across the upcrossing interval over any chain of 1's flanked on either side by 0's (for instance, consider the chain of 1's stretching from $\tau = 2$ to $\tau = 4$ in Figure 1). It may be noted that our definition of $K_\tau(\omega)$ does not correspond to those in standard probability texts (for instance, Billingsley [5, Theorem 35.3]) where a sequence of 1's also indicates the *duration* of an upcrossing. We, on the contrary, are not interested in the duration (since it is fixed at one period by construction) but rather in the *number* of upcrossings over the given interval (since each is indicative of more than ϵ-divergence in capital stocks); for the same reason, as opposed to martingale theory, we do not wish to count downcrossings over any interval.

To count the number of upcrossings given $\omega \in \Omega$, for each $\tau \in \mathcal{I}_+$ let:

$$Z_\tau^{\alpha,\beta}(\omega) = \begin{cases} 1, & K_\tau^{\alpha,\beta}(\omega) = 1,\ K_{\tau+1}^{\alpha,\beta}(\omega) = 0 \\ 0, & \text{otherwise} \end{cases}$$

Given an arbitrary planning horizon $N \in \mathcal{I}_+$ and state ω, let $U_N^\epsilon(\omega) = \sum_{\tau=1}^{N} Z_\tau^{\alpha,\beta}(\omega)$ represent the total number of upcrossings over $[\alpha, 2\mu\beta+\alpha]$ of $\langle V_\tau^{\alpha,\beta}(\omega)\rangle$ on the time-line τ. Our version of the Martingale Upcrossing result states:

Lemma 6 *For any $\epsilon > 0$ and $N \in \mathcal{I}_+$, the expected number of upcrossings across the interval $[\alpha, 2\mu\beta+\alpha]$ of the value-loss process $\langle V_\tau^{\alpha,\beta}\rangle$ over a planning horizon of length $N+1$ is bounded from above independently of N.*

With the help of Lemma 6, we can prove the late turnpike property as follows. Since an upcrossing over the given interval occurs if and only if there is more than ϵ-divergence in capital stocks, and this in turn transpires only for even-valued τ (i.e. our original time-line), Lemma 6 implies $2\mu\beta \sum_{t=1}^{N} \nu(S_t^\epsilon) \leq EV_N^{\alpha,\beta}$. Since value-loss is bounded in expectation independently of N, a passage to the limit as $N \longrightarrow \infty$ yields $\sum_{t=1}^{\infty} \nu(S_t^\epsilon) < \infty$. The first Borel-Cantelli lemma (Billingsley [5, Theorem 4.3]) then ensures that the set of realizations on which capital stocks diverge by more than ϵ infinitely often has zero probability measure. Hence, we have a resolution of the late turnpike property without the invocation of any strong uniformity restriction.

Proposition 3 *Consider initial stocks $s, y \in \mathcal{L}$, $s \leq y$ almost surely such that y is reachable from s. Under Assumptions* (A.1)-(A.4), $||k_t^s - k_t^y|| \longrightarrow 0$ *almost surely as $t \longrightarrow \infty$.*

6 Conclusion

In this paper, we have exploited the underlying stochastic primitive to obtain the late turnpike property in convex aggregate growth models without imposing any strong uniformity restriction. In this regard, our paper echoes the argument put forward in Amir [1] that growth models under uncertainty should not be simply extensions of the deterministic case with the stochastic element as a mere addendum. Rather, uncertainty should add in an essential way to the results derivable from the certainty case. A similar consideration had also motivated Chang [12] to put forth a expected value-loss assumption conditioned to the particular dictates of a stochastic paradigm.

There remains the issue of whether our technique can be extended to *multisector* optimal growth models. As noted earlier, such models can display complicated dynamics for low values of the discount factor. However, even for discount factors sufficiently close to unity, some form of uniformity is generally invoked to obtain the late turnpike property (Guerrero-Luchtenberg [16], McKenzie [22], Montrucchio [28]). It is an interesting question, therefore, whether critical manipulation of the stochastic environment, along with a suitable restriction of reachability on initial stocks, permits a derivation of the late turnpike property for sufficiently high values of the discount factor without recourse to strong uniformity restrictions.

Appendix

Proof of Lemma 1 Note first of all that:

$$\begin{aligned}\mathcal{B}_t &\subseteq \sigma(\mathcal{H}_0) \times \sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_t) \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots \\ &= \mathcal{E}_0 \times \mathcal{E}_1 \times \cdots \times \mathcal{E}_t \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots \\ &\subseteq \sigma\left(\mathcal{E}_0 \times \mathcal{E}_1 \times \cdots \times \mathcal{E}_t \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots\right) = \mathcal{F}_t\end{aligned}$$

It, therefore, follows that $\sigma(\mathcal{B}_t) \subseteq \mathcal{F}_t$. To prove the converse, we use a result from Yeh [33, Lemma 1.3] that $\sigma(\mathcal{H}_0) \times \sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_t) \subseteq \sigma(\mathcal{H}_0 \times \mathcal{H}_1 \times \cdots \times \mathcal{H}_t)$. Using this result:

$$\begin{aligned}&\sigma(\mathcal{H}_0) \times \sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_t) \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots \\ &\qquad \subseteq \sigma\left(\mathcal{H}_0 \times \mathcal{H}_1 \times \cdots \times \mathcal{H}_t\right) \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots \\ &\qquad \subseteq \sigma\left(\mathcal{H}_0 \times \mathcal{H}_1 \times \cdots \times \mathcal{H}_t \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots\right) = \sigma(\mathcal{B}_t)\end{aligned}$$

It now follows that $\mathcal{F}_t = \sigma\left(\sigma(\mathcal{H}_0) \times \sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_t) \times \Omega_{t+1} \times \Omega_{t+2} \times \cdots\right) \subseteq \sigma(\mathcal{B}_t)$.

For $i = 0, 1, \ldots, t$, since Ω_i is open and can be expressed as the countable union of the elements of $\mathcal{H}_i$, it follows that $\mathcal{B}_t$ covers Ω. To show that $\mathcal{B}_t$ can be assumed to be a countable collection of *mutually disjoint* sets, define one possible countable partition, $\mathcal{B}'_t = \langle B'_{n,t} \rangle$, as $B'_{0,t} = B_{0,t}$, $B'_{n,t} = B_{n,t} \setminus \cup_{j=0}^{n-1} B_{j,t}$ for $n \in \mathcal{I}_+ \setminus \{0\}$. Let $\mathcal{F}'_t$ denote the σ-field generated by $\mathcal{B}'_t$. For each $\omega \in \Omega$, and a set $B_{n,t} \in \mathcal{B}_t$ containing ω, by construction there is a set $B'_{n,t} \in \mathcal{B}'_t$ such that $\omega \in B'_{n,t} \subseteq B_{n,t}$. Hence, it follows that $\mathcal{F}_t \subseteq \mathcal{F}'_t$. Conversely, because a σ-field is closed under set-theoretic difference (given closure under complementation and finite intersections) and countable unions, we have $\mathcal{B}'_t \subseteq \mathcal{F}_t$. Hence, $\mathcal{F}'_t = \sigma(\mathcal{B}'_t) \subseteq \mathcal{F}_t$. $\triangle$

Proof of Lemma 3 Suppose there exists $F \in \mathcal{F}_0$, $\nu(F) > 0$, where $F \subset \Delta(s, y)$ but $F \cap (\Omega \setminus S_0) = \emptyset$. Then $s(\omega) = y(\omega)$ but $k_0^y(\omega) \neq k_0^s(\omega)$ for all $\omega \in F$. From Lemma 2, this implies $k_0^y(\omega) > k_0^s(\omega)$ on F. From the Euler equations for $i = s, y$:

$$\int_F u'_0(c_0^i)d\nu = \int_F u'_1(c_1^i) f'_0(k_0^i, \omega_1) d\nu \tag{15}$$

Since $c_0^s(\omega) = s(\omega) - k_0^s(\omega) > y(\omega) - k_0^y(\omega) = c_0^y(\omega)$ on F, it follows from (A.3) that:

$$\int_F u'_1(c_1^s) f'_0(k_0^s, \omega_1) d\nu < \int_F u'_1(c_1^y) f'_0(k_0^y, \omega_1) d\nu \tag{16}$$

On the other hand, from (A.2), $f_0'(k_0^s, \omega_1) \geq f_0'(k_0^y, \omega_1)$ almost surely. Further, $\langle(k_t^i, c_t^i)\rangle$ is also optimal starting from initial stock $f_0(k_0^i, \omega_1)$ for otherwise its optimality will be contradicted. From Lemma 2 this implies $c_1^y \geq c_1^s$ almost surely. Using (A.3), it follows that:

$$\int_F u_1'(c_1^s) f_0'(k_0^s, \omega_1) d\nu \;\geq\; \int_F u_1'(c_1^y) f_0'(k_0^y, \omega_1) d\nu \tag{17}$$

However, (16) and (17) are mutually contradictory. Now, for any $t \in \mathcal{I}_+ \backslash 0$, suppose there exists $F_t \in \mathcal{F}_t$, $\nu(F_t) > 0$, where $F_t \subset \Omega \backslash S_{t-1}$ but $F_t \cap (\Omega \backslash S_t) = \emptyset$. Repeating the above argument once again generates a contradiction. △

Proof of Lemma 4 Consider the partition, $\mathcal{B}_t$, of Ω, $\mathcal{F}_t = \sigma(\mathcal{B}_t)$. Recalling (6), $\mathcal{B}_t = \mathcal{B}_t' \cup \mathcal{B}_t''$. Since $\mathcal{B}_t$ covers Ω, the sub-collection $\mathcal{B}_t'$ is non-empty. Without loss of generality, it can be assumed that $\mathcal{B}_t'$ and $\mathcal{B}_t''$ are countably infinite.[10] We now define a partition, $\mathcal{C}_t = \langle C_{n,t} \rangle$, of Ω by letting $C_{n,t} = B_{n,t}' \cup B_{n,t}''$, $n \in \mathcal{I}_+$. By construction, $\nu((\Omega \backslash S_{t+1}^\epsilon) \cap C_{n,t}) > 0$ for all $C_{n,t} \in \mathcal{C}_t$. Since $\mathcal{C}_t$ is a coarser partition of Ω than $\mathcal{B}_t$, $\sigma(\mathcal{C}_t) \subseteq \mathcal{F}_t$. △

Proof of Lemma 5 Since each $D_{n,t} \in \mathcal{D}_t$ is a finite union of sets in $\mathcal{B}_t$, it follows that $\mathcal{D}_t \subset \mathcal{F}_t$ and, therefore, $\sigma(\mathcal{D}_t) \subseteq \mathcal{F}_t$. Conversely, it can be verified that each $B_{n,t} \in \mathcal{B}_t$ can be derived from $\mathcal{D}_t$ through the operations of finite intersection, finite union, and set-theoretic difference.[11] Therefore, $\mathcal{B}_t \subset \sigma(\mathcal{D}_t)$, and hence, $\mathcal{F}_t \subseteq \sigma(\mathcal{D}_t)$. △

Proof of Proposition 1 The proof is provided for $\langle X_\tau \rangle$ and is identical for $\langle Y_\tau \rangle$. $\langle \mathcal{F}_\tau \rangle$-adaptability follows by construction. From Billingsley [5, Section 35], to establish the martingale property we need to demonstrate that for any $F \in \mathcal{F}_\tau$:

$$\int_F X_{\tau+1}\, d\nu \;=\; \int_F X_\tau\, d\nu \;=\; \mu\nu(F) \tag{18}$$

The last equality follows from the fact that $\langle X_\tau \rangle$ at the "original" time-periods (i.e. even values of τ) is flanked by the constant value of μ at the "intermediate" time-periods (i.e. odd values of τ). We establish this result by means of Dynkin's $\pi - \lambda$ theorem. Note that (18) is trivially true if $\nu(F) = 0$. Therefore, without loss of generality, we restrict attention to $\mathcal{F}_\tau$-measurable sets of strictly positive measure. We will also let $\tau + 1$ be even-valued so

[10] As subsets of a compact metric space, S_t^ϵ and $(\Omega \backslash S_t^\epsilon)$ are Lindelöf spaces (Munkres [29, Ch.4]) and, therefore, can be covered by a countable collection of open sets, $S_t^\epsilon = \cup_{n=1}^\infty O_{n,t}''$ and $\Omega \backslash S_t^\epsilon = \cup_{n=1}^\infty O_{n,t}'$. $\mathcal{B}_t'$ and $\mathcal{B}_t''$ can be taken to the basis sets generating $\langle O_{n,t}' \rangle$ and $\langle O_{n,t}'' \rangle$ respectively.

[11] For instance, suppose $D_{1,t} = B_{1,t}' \cup B_{1,t}'' \cup B_{2,t}''$. By construction, there are distinct sets, $D_{m,t}, D_{n,t} \in \mathcal{D}_t$, of the form $D_{m,t} = B_{m,t}' \cup B_{2,t}'' \cup B_{m,t}''$ and $D_{n,t} = B_{n,t}' \cup B_{1,t}'' \cup B_{n,t}''$. Therefore, $B_{2,t}'' = D_{1,t} \cap D_{m,t}$, $B_{1,t}'' = D_{1,t} \cap D_{n,t}$, and $B_{1,t}' = D_{1,t} \backslash \{B_{1,t}'' \cup B_{2,t}''\}$.

that it corresponds to the original time-line (the same argument applies with τ even-valued, so once again there is no loss of generality stemming from this assumption).

We start with the collection $\mathcal{D}_\tau$. Taking the expectation of $X_{\tau+1}$ over $D_{n,\tau} = C_{n,\tau} \cup B''_{n,\tau}$:

$$\int_{D_{n,\tau}} X_{\tau+1}\, d\nu = \int_{S^{\epsilon}_{\tau+1} \cap D_{n,\tau}} M^{s}_{\tau+1}\, d\nu + \int_{(\Omega \backslash S^{\epsilon}_{\tau+1}) \cap D_{n,\tau}} \tilde{\zeta}_{\tau+1} \delta_{\tau+1} d\nu \tag{19}$$

By construction, $C_{n,\tau} \cap B''_{n,\tau} = \emptyset$ and $\nu((\Omega \backslash S^{\epsilon}_{\tau+1}) \cap B''_{n,\tau}) = 0$. Therefore, the extreme term on the right in (19) becomes:

$$\int_{(\Omega \backslash S^{\epsilon}_{\tau+1}) \cap D_{n,\tau}} \tilde{\zeta}_{\tau+1} \delta_{\tau+1} d\nu = \int_{(\Omega \backslash S^{\epsilon}_{\tau+1}) \cap C_{n,\tau}} \tilde{\zeta}_{\tau+1} \delta_{\tau+1} d\nu \tag{20}$$

Substituting (20) in (19), and noting that by definition $\tilde{\zeta}_{\tau+1}$ assumes the constant value of $\zeta_{\tau+1}(D_{n,\tau})$ on $C_{n,\tau}$, we have:

$$\int_{D_{n,\tau}} X_{\tau+1}\, d\nu = \int_{S^{\epsilon}_{\tau+1} \cap D_{n,\tau}} M^{s}_{\tau+1}\, d\nu + \zeta_{\tau+1}(D_{n,\tau}) \int_{(\Omega \backslash S^{\epsilon}_{\tau+1}) \cap C_{n,\tau}} \delta_{\tau+1} d\nu \tag{21}$$

Substituting for $\zeta_{\tau+1}(D_{n,\tau})$ from (8) in (21), we observe that (18) is satisfied. Now let $\mathcal{O}_\tau$ denote the collection of sets which are a countable union of *disjoint* sets in $\mathcal{D}_\tau$. For any $O_{k,\tau} = \cup_{j=1}^{\infty} D_{j,\tau} \in \mathcal{O}_\tau$, $k \in \mathcal{I}_+$:

$$\int_{O_{k,\tau}} X_{\tau+1}\, d\nu = \int_{\cup_{j=1}^{\infty} D_{j,\tau}} X_{\tau+1}\, d\nu = \sum_{j=1}^{\infty} \int_{D_{j,\tau}} X_{\tau+1}\, d\nu = \sum_{j=1}^{\infty} \int_{D_{j,\tau}} X_{\tau}\, d\nu = \int_{\cup_{j=1}^{\infty} D_{j,\tau}} X_{\tau}\, d\nu = \int_{O_{k,\tau}} X_{\tau}\, d\nu \tag{22}$$

Therefore, (18) holds for all sets in $\mathcal{O}_\tau$. Also note that $\mathcal{O}_\tau$ constitutes a π-system, i.e. it is closed under finite intersections. Given any $O_{k,\tau}, O_{m,\tau} \in \mathcal{O}_\tau$, one possibility is $O_{k,\tau} \cap O_{m,\tau}$ is empty, in which case (18) holds trivially. The other possibility is that $O_{k,\tau} \cap O_{m,\tau}$ is non-empty, and is, therefore, a countable collection of disjoint sets in $\mathcal{D}_\tau$; in this case a string of equalities as in (22) establishes (18).

Now consider the collection, Λ, of all $\mathcal{F}$-measurable subsets of Ω for which (18) holds. We now show that Λ constitutes a λ-system. First of all note that $\Omega \in \Lambda$, given that Ω is open in the product topology which generates $\mathcal{F}$. Second, Λ is closed under proper differences. To verify, consider $F, G \in \Lambda$ such that $G \subset F$. Since $F = (F\backslash G) \cup G$, the subtractive property of probability measures yields:

$$\int_{F\backslash G} X_{\tau+1}\, d\nu \;=\; \int_F X_{\tau+1}\, d\nu - \int_G X_{\tau+1}\, d\nu \;=\; \mu[\nu(F) - \nu(g)] \;=\; \mu\nu(F\backslash G) \tag{23}$$

Third, Λ is closed under monotone limits. To verify, consider a sequence $\langle F_n \rangle$ of $\mathcal{F}$-measurable sets and $F \in \mathcal{F}$ such that $F_n \uparrow F$. Then $0 \leq \chi(F_n) \uparrow \chi(F)$.[12] Using the Monotone Convergence theorem and the sequential continuity from below of probability measures (Billingsley [5, Theorem 16.2, Theorem 2.1]), it follows that:

$$\int_F X_{\tau+1}\, d\nu \;=\; \lim_{n\to\infty} \int_{F_n} X_{\tau+1}\, d\nu \;=\; -\lim_{n\to\infty} \mu\nu(F_n) \;=\; \mu\nu(F) \tag{24}$$

This indicates closure under monotone limits. Note that $\mathcal{O}_\tau \subset \Lambda$ since sets in $\mathcal{O}_\tau$ satisfy (18) and are $\mathcal{F}$-measurable. From Dynkin's $\pi - \lambda$ theorem, $\sigma(\mathcal{O}_\tau) \subseteq \Lambda$. The martingale property now follows, since an argument similar to that in Lemma 5 establishes that $\sigma(\mathcal{O}_\tau) = \sigma(\mathcal{D}_\tau) = \mathcal{F}_\tau$. To establish the bound on the expectation of $\langle X_\tau \rangle$, note that since $\Omega \in \mathcal{F}_\tau$, we have $\int_\Omega X_{\tau+1}\, d\nu = \mu\nu(\Omega) = \mu$. $\triangle$

Proof of Proposition 2 Let the value-loss process $\langle V_\tau \rangle$ be defined as:

$$V_\tau \;=\; X_\tau + Y_\tau\,, \quad \tau \in \mathcal{I}_+\backslash\{0\} \tag{25}$$

with the initial value, $V_0 = 0$.[13] If τ is odd, then by construction, $V_\tau = 0$. Now suppose that τ is even. If $\omega \in S^\epsilon_\tau$, then $V_\tau(\omega) = M^s_\tau(\omega) + M^y_\tau(\omega) > 2\mu$. On the other hand, if $\omega \in (\Omega\backslash S^\epsilon_\tau)$, then $V_\tau(\omega) = \delta_\tau(\omega)[\tilde{\zeta}_\tau(\omega) + \tilde{\xi}_\tau(\omega)] < 0$. Further, as the sum of $\langle \mathcal{F}_\tau \rangle$-martingales $\langle X_\tau \rangle$ and $\langle Y_\tau \rangle$ with means μ and $-\mu$ respectively, $\langle V_\tau \rangle$ is a zero-mean $\langle \mathcal{F}_\tau \rangle$-martingale. $\triangle$

Proof of Lemma 6 Consider any $\langle V^{\alpha,\beta}_\tau \rangle$, $\alpha, \beta \in \mathcal{R}_{++}$. For arbitrary $N \in \mathcal{I}_+$ and any $\omega \in \Omega$, on the time-line τ:

$$V^{\alpha,\beta}_N \;\geq\; V^{\alpha,\beta}_N - V^{\alpha,\beta}_0$$

[12] Fix $\epsilon > 0$. If $\omega \notin F$, then $\omega \notin F_n$ for all n and hence $\chi(F) - \chi(F_n) = 0 < \epsilon$ for all n. If $\omega \in F$, then there exists $m \in \mathcal{I}_+$ such that $\omega \in F_n$ for all $n \geq m$ and hence $\chi(F) - \chi(F_n) = 0 < \epsilon$ for all $n \geq m$.

[13] Since our time line starts at zero, fixing the initial value $V_0 = 0$ implies that we ignore a potential deviation in capital stocks by more than ϵ in the initial period. But ignoring *one* possible deviation in period 0 is inconsequential given our concern with the *long run* behaviour of optimal programs.

$$
\begin{aligned}
&= \sum_{\tau=1}^{N} \left(V_{\tau}^{\alpha,\beta} - V_{\tau-1}^{\alpha,\beta} \right) \\
&= \sum_{\tau=1}^{N} K_{\tau}^{\alpha,\beta} \left(V_{\tau}^{\alpha,\beta} - V_{\tau-1}^{\alpha,\beta} \right) + \sum_{\tau=1}^{N} \left(1 - K_{\tau}^{\alpha,\beta} \right) \left(V_{\tau}^{\alpha,\beta} - V_{\tau-1}^{\alpha,\beta} \right)
\end{aligned}
\tag{26}
$$

The reference to the argument, ω, is suppressed for notational ease. Now consider an up-crossing represented by a chain of 1's flanked by 0's: $K_{m}^{\alpha,\beta} = 0$, $K_{m+1}^{\alpha,\beta} = \cdots = K_{n}^{\alpha,\beta} = 1$, $K_{n+1}^{\alpha,\beta} = 0$. Note by construction that $V_{n}^{\alpha,\beta} \geq 2\mu\beta + \alpha$ and $V_{m}^{\alpha,\beta} = \alpha$ (where the latter follows from the fact that, with the exception of $\tau = 0$, $K_{\tau}^{\alpha,\beta} = 0$ only for odd-valued τ which succeed an upcrossing and never at any even-valued τ; further, at any odd-valued τ, value-loss takes the constant value of α). Therefore:

$$
\sum_{\tau=m+1}^{n} K_{\tau}^{\alpha,\beta} \left(V_{\tau}^{\alpha,\beta} - V_{\tau-1}^{\alpha,\beta} \right) = V_{n}^{\alpha,\beta} - V_{m}^{\alpha,\beta} \geq 2\mu\beta \tag{27}
$$

It now follows that if the number of upcrossings are equal to U_{N}^{ϵ} for the given ω, then:

$$
\sum_{\tau=1}^{N} K_{\tau}^{\alpha,\beta} \left(V_{\tau}^{\alpha,\beta} - V_{\tau-1}^{\alpha,\beta} \right) \geq 2\mu\beta U_{N}^{\epsilon} \tag{28}
$$

Also note from the martingale property of $\langle V_{\tau}^{\alpha,\beta} \rangle$ and the predictability of $K_{\tau}^{\alpha,\beta}$ that:

$$
\begin{aligned}
&\sum_{\tau=1}^{N} \int \left(1 - K_{\tau}^{\alpha,\beta} \right) \left(V_{\tau}^{\alpha,\beta} - V_{\tau-1}^{\alpha,\beta} \right) d\nu = \\
&\qquad \sum_{\tau=1}^{N} \int \left(1 - K_{\tau}^{\alpha,\beta} \right) E \left[V_{\tau}^{\alpha,\beta} - V_{\tau-1}^{\alpha,\beta} \, || \, \mathcal{F}_{t-1} \right] d\nu = 0
\end{aligned}
\tag{29}
$$

Combining (26), (28) and (29) yields:

$$
2\mu\beta \int U_{N}^{\epsilon} d\nu \leq \int V_{N}^{\alpha,\beta} d\nu \tag{30}
$$

The result follows by noting that the expectation of $V_{N}^{\alpha,\beta}$ is bounded from above by α. $\triangle$

Proof of Proposition 3 Revert to the original time-line (i.e. even-valued τ) since the variable $Z_t^{\alpha,\beta} = 1$ only in time periods $t = 1, 2, \ldots$, and not in periods $t = \frac{1}{2}, 1\frac{1}{2}, \ldots$. Further, note that an upcrossing takes place over $[\alpha, 2\mu\beta + \alpha]$ if and only if there is more than ϵ-divergence in capital stocks, i.e. $Z_t^{\alpha,\beta} = 1$ if and only if $\chi(S_t^\epsilon) = 1$. From (30), and the fact that $EU_N^\epsilon = E\sum_{t=1}^N Z_t^{\alpha,\beta} = \sum_{t=1}^N \nu(S_t^\epsilon)$, we can conclude that:

$$\begin{aligned} 2\mu\beta \sum_{t=1}^{N} \nu(S_t^\epsilon) &= 2\mu\beta \int \sum_{t=1}^{N} \chi(S_t^\epsilon) d\nu = \\ 2\mu\beta \int \sum_{t=1}^{N} Z_t^{\alpha,\beta} d\nu &= 2\mu\beta \int U_N^\epsilon d\nu \leq \int V_N^{\alpha,\beta} d\nu \end{aligned} \tag{31}$$

Since the expectation of $V_N^{\alpha,\beta}$ is bounded from above independently of N, letting $N \to \infty$ in (31) yields $\sum_{t=1}^{\infty} \nu(S_t^\epsilon) < \infty$. An application of the first Borel-Cantelli lemma then yields $\nu(\{\omega : \limsup_{t\to\infty} S_t^\epsilon\}) = 0$. $\triangle$

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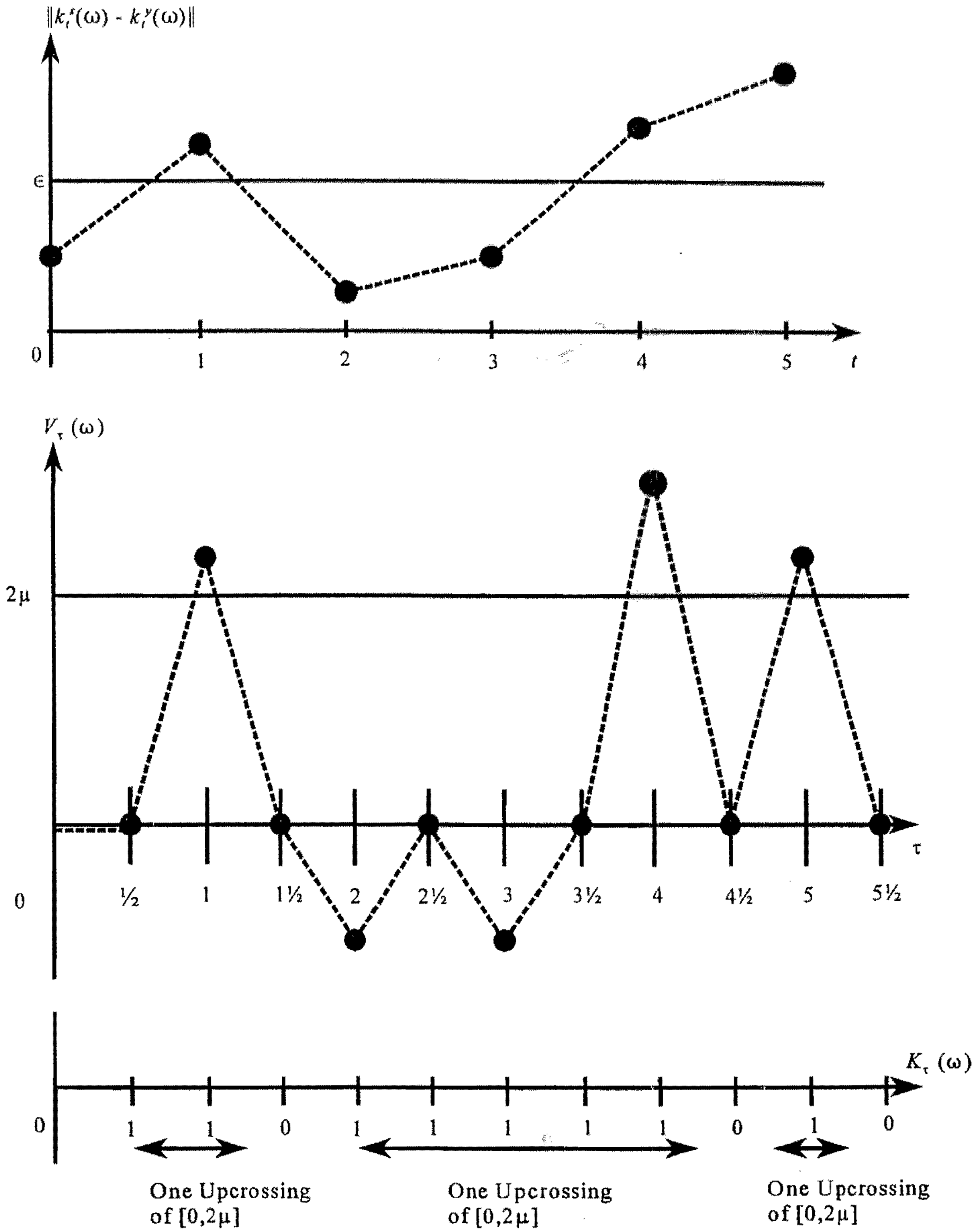

Figure 1: Tracking Divergence in Capital Stocks Through Upcrossings of the Value-Loss Process

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