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Sanyyam Khurana[*]

Abstract

In this paper, we characterize all the Bayesian equilibria of a first-price auction for asymmetric bidders with risk averse preferences. The necessary conditions for an equilibrium are pure strategy, continuity and strict monotonicity. Next, we show that first-order stochastic dominance is a necessary condition and conditional stochastic dominance is a sufficient condition to unambiguously rank the bidding strategies. We establish bidders' preferences for the first-price and the second-price auction under different types of risk aversion. Finally, for a special family of utility functions and distribution functions, we study the impact of asymmetry on seller's revenue in a first-price auction.

JEL classification: D44, D82
Keywords: Asymmetric auctions, first-price auction, risk aversion

1 Introduction

This paper provides necessary and sufficient conditions of an equilibrium in a first-price auction game with incomplete information when bidders are asymmetric and have risk averse preferences. The assumption of symmetric bidders and risk neutral preferences in auction theory is too strong. Asymmetric auctions with risk neutral preferences have been extensively studied in Plum [19]; Lebrun [10]; Maskin and Riley [13, 14, 15]; Cheng [3]; Cantillon [2] and Kirkegaard [5]. On the other hand, symmetric auctions with risk averse preferences have been studied in Holt [4]; Riley and Samuelson [20]; Maskin and Riley [12]; and Matthews [17]. In this paper, we study asymmetric auctions with risk averse preferences.

Consider two bidders playing an auction game with incomplete information for an indivisible object. Nature draws a type for each bidder and informs them privately. The probability distributions of bidders' types are independently distributed and are common knowledge among bidders and the seller. The von-Neumann-Morgenstern utility function of every bidder is continuous, strictly increasing and strictly concave on the real line. Bidders are

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homogeneous in utility functions but heterogeneous in distribution functions. The bidder with the highest bid wins the object and pays according to the underlying auction mechanism.

An explicit expression for bidding strategy in a first-price auction does not exist when bidders are asymmetric or risk averse.[1] Moreover, it is not always possible to find analytical solutions for a given family of distribution functions when bidders are asymmetric. Nevertheless, the differential equations characterizing the bidding strategies are quite useful in examining the properties of equilibrium such as bidding behavior, revenue rankings, bidders' preferences, etc. In order to characterize the bidding strategy, some structure on the smoothness of the bidding functions is required which includes measurability, monotonicity and continuity. A priori, assuming that bidding strategies are measurable functions, monotonically increasing and continuous restricts the class of bidding strategy. Lebrun [10] characterizes all the Bayesian equilibria of a first-price auction by making no prior assumption on the bidding strategy. However, he assumes that bidders are risk neutral. In this paper, we characterize all the Bayesian equilibria of a first-price auction with asymmetric and risk averse bidders. While characterizing the equilibria, we do not presume that bidding strategies are pure, strictly increasing and continuous. Thus, there is a possibility for a random strategy equilibrium which may be discontinuous and non-increasing. Theorem 1 of this paper shows that the necessary conditions characterizing the equilibria are pure strategy, strict monotonicity and continuity. Furthermore, the differential equations representing the bidding strategies are both necessary and sufficient.

Stochastic orders play a crucial role in comparing bidding strategies and bid distributions. Some of the stochastic orders which are widely used in auction theory are first-order stochastic dominance, second-order stochastic dominance, hazard rate dominance, conditional stochastic dominance (or reverse hazard rate dominance) and likelihood ratio dominance. It is well known that, with risk neutral bidders, conditional stochastic dominance is a sufficient condition to unambiguously rank the bidding strategy and the bid distribution (Lebrun [10] and Maskin and Riley [13]). Kirkegaard [5] studies the bidding behavior by not restricting to stochastic orders. In this paper, we follow the approach used by Kirkegaard [5]. We allow (a) the distribution functions to intersect a finite number of times, and (b) the slope of the ratio of distribution functions to change its sign a finite number of times. The special cases of (a) and (b) include first-order stochastic dominance and conditional stochastic dominance respectively. In Theorem 2, we derive an upper bound on the number of times the expected utility functions intersect. In Theorem 3, we derive an upper bound on the number of times the ratio of expected utility functions and the ratio of distribution functions intersect. The implications of these results convey that (a) first-order stochastic dominance is a necessary condition and conditional stochastic dominance is a sufficient

[1] For example, see [19], [12].

condition to unambiguously rank the bidding strategies, and (b) first-order stochastic dominance is a sufficient condition to unambiguously rank the bid distributions.

In section 5 and henceforth, we consider that one bidder is "strong" and the other is "weak". Roughly speaking, a bidder is strong or weak in the sense that the probability of getting a high type is more for the strong bidder than the weak bidder. We compare bidders' preferences for the first-price and the second-price auction under different types of risk aversion. It is known that, when bidders are symmetric and have constant absolute risk averse utility function, then bidders are indifferent between the first-price and the second-price auction (Matthews [17]). We show that the above result no longer holds when bidders are asymmetric. It is also known that, when bidders are symmetric and have risk neutral preferences, then the weak bidder prefers a first-price auction and the strong bidder prefers a second-price auction (Maskin and Riley [13]). We show that the above result cannot be generalized for different types of risk aversion whenever bidders are asymmetric.

An important question to ask is the following: Does the seller prefer asymmetric bidders over symmetric bidders in a first-price auction? The answer depends on the format of auction mechanism, how the symmetric bidders are related to asymmetric bidders, family of distribution functions, family of utility functions and so on. For a special family of distribution functions and increasing absolute risk averse utility function, we compare the seller's revenue for different degrees of asymmetry in a first-price auction. We show that the seller's revenue declines as the degree of asymmetry is high. Another important question that arises is that which auction format is preferred by the seller when bidders are asymmetric and risk averse? The revenues generated from first-price and second-price auction are equivalent whenever bidders are symmetric and risk neutral. However, whenever bidders are asymmetric, the revenues from first-price and second-price auction cannot be "generally" compared. Nevertheless, a first-price auction is revenue superior to a second-price auction under restrictive conditions which includes reverse hazard rate dominance and "stretched" or "shifted" probability distributions. Unfortunately, this revenue ranking does not hold if bidders are risk averse. Nevertheless, by restricting attention to linear bidding strategies and a special family of utility functions and distribution functions, we show that a first-price auction is revenue superior to a second-price auction.

1.1 The literature

In auction literature, the key assumptions are (a) symmetric bidders, (b) independent and private types, (c) risk neutrality, (d) no collusion, and (e) the bidder with highest bid wins the object. Myerson [18] and Riley and Samuelson [20] independently show that revenue equivalence theorem holds if all the assumptions (a)-(e) are satisfied. Notice that the revenue equivalence theorem does not depend on the nature of payment structure. This means that the theorem holds for all types of auction mechanisms such as first-price auction, second-price auction, third-price auction, all-pay auction and so on. Moreover, both first-price and second-price auction mechanisms are efficient, i.e., they award the object to the bidder with the highest type (or valuation). Another important feature is that an explicit expression exists for the bidding strategy in a first-price auction. However, most of these results do not hold if any of the above assumption is violated.

Riley and Samuelson [20]; Maskin and Riley [16]; and Matthews [17] relax the assumption of risk neutrality. Riley and Samuelson [20] and Maskin and Riley [16] independently show that revenue equivalence theorem does not hold. Specifically, they show that a first-price auction is revenue superior to a second-price auction. Also, both first-price and second-price auction mechanisms are efficient. On the other hand, Matthews [17] compares bidders' preferences under different types of absolute risk aversion. He shows that, with increasing absolute risk aversion, bidders prefer a first-price auction; with constant absolute risk aversion, they are indifferent; and with decreasing absolute risk aversion, they prefer a second-price auction.

Plum [19]; Lebrun [9, 10]; Maskin and Riley [13]; Kirkegaard [5]; Cheng [3]; and Cantillon [2] relax the assumption of symmetric bidders. Plum [19] and Lebrun [10] characterizes all the Bayesian equilibria of a first-price auction. While characterizing the equilibria, Plum [19] presumes that bidding strategies are pure. He shows that bidding strategies are strictly increasing and continuous. On the other hand, by making no prior assumption on the bidding strategies, Lebrun [10] characterizes all the Bayesian equilibria. He shows that bidding strategies are pure, strictly increasing and continuous. Maskin and Riley [13] (henceforth, M-R) studies bidding behavior and other properties of equilibrium by considering one strong and one weak bidder in their analysis. They show that the weak bidder bids more aggressively and produces a weaker bid distribution than the strong bidder. Moreover, the first-price auction is inefficient, i.e., the bidder with the lowest valuation also wins the auction with positive probability. On the other hand, the second-price auction is efficient, since it still remains a weakly dominant strategy to bid your own type. M-R also shows that general revenue rankings for the first-price and the second-price auction cannot be established. But by making certain assumptions on the asymmetry of bidders, they show that a first-price auction generates more revenue for the seller as compared to a second-price auction. M-R also compares the bidders' preferences for the first-price and the second-price auction. They show that the weak bidder prefers a first-price auction whereas the strong bidder prefers a second-price auction. Lebrun [9] studies the bidding behavior when the distribution function of a bidder changes stochastically. He shows that a bidder starts behaving more aggressively due to a stochastic improvement in the distribution function of the other bidder. Kirkegaard [5] studies the bidding behavior by not restricting to stochastic orders. He shows that conditional stochastic dominance is a sufficient condition and first-order stochastic dominance is a necessary condition to unambiguously compare the bidding strategies. For a special family of distribution functions and linear bidding strategies, Cheng [3] shows that a first-price auction is revenue superior to a second-price auction. Cantillon [2] compares the seller's expected revenue for symmetric and asymmetric bidders. For three special family of distribution functions, she proves that the seller prefers symmetric bidders over asymmetric bidders.

In this paper, we relax the assumption of symmetric bidders and risk neutrality simultaneously. We study the impact of asymmetry and risk aversion on various properties of bidding behavior.

1.2 Outline

The structure of the paper is as follows. In section 2, we characterize all the Bayesian equilibria of a first-price auction when the bidders are risk averse expected utility maximizers. In section 3, we compare the bidding strategies by not restricting to stochastic orders. In section 4, we study the bidding behavior when the distribution function of a bidder changes stochastically. In section 5, we establish bidders' preferences under different types of risk aversion. In section 6, we study the impact of asymmetry on seller's revenue for a special family of utility functions and probability distributions. In section 7, we compare seller's revenue for a special family of utility and probability distributions. Section 8 concludes the paper.

2 Economic environment

We introduce the formalism of the model. Consider the following independent private valuation model. A single unit of an indivisible object is available for sale through a first-price sealed bid auction. There are two asymmetric bidders with risk averse preferences. The set of bidders is denoted by $N = \{1, 2\}$. Let $T_i = [0, a_i] \subset \Re$ be the type space of bidder i and $t_i \in T_i$ be his type. Let $T = T_1 \times T_2$ be the product type space and $\boldsymbol{t} \in T$ be a type profile. Nature draws a type profile $\boldsymbol{t} \in T$ and privately informs t_i to bidder i, i.e., bidder i knows t_i and not t_j for every $j \neq i$. Let $B_i \subseteq \Re_+$ be the bidding space of bidder i and $b_i \in B_i$ be his bid. Let $B = B_1 \times B_2$ be the product bidding space and $\boldsymbol{b} \in B$ be a bidding profile. The von-Neumann-Morgenstern utility function for both the bidders is $u : \Re_+ \to \Re$ with $u(0) = 0$, $u' > 0$ and $u'' < 0$.[2] The distribution function is given by $F_i : T_i \to \Re$. We assume that F_i is twice continuously differentiable. We also assume that the

[2]Alternatively, in a more general framework, $u_i = u(b, t_i)$ such that u_i is monotonic and weakly supermodular. In our framework, the assumption of strictly increasing and concave is equivalent to the assumption of monotonicity and weak supermodularity in a more general framework.

distribution function has positive density, which is denoted by f_i, and the density function is always bounded away from zero, i.e., $f_i > 0$ for every $t_i \in T_i$. Moreover, the distribution functions are independently distributed and are common knowledge among bidders and the seller.

The structure of the game is as follows. The seller of the object conducts a first-price auction. All the bidders submit their bids simultaneously in a sealed envelope. The bidder with the highest bid is the winner of the auction and pays his own bid whereas the loser of the auction does not pay anything. In case of a tie, the seller chooses the winner by a fair lottery. All the bidders are expected utility maximizers. For simplicity, we assume that the seller is risk neutral and his reservation utility is zero.

The payoff function $\pi_i : T_i \times B \to \Re$ for bidder i is defined as

$$\pi_i(t_i, b_i, b_j) = \begin{cases} \frac{u(t_i - b_i)}{|Z|} & ; i \in Z \\ u(0) & ; i \notin Z \end{cases}$$

where $Z = \{i | \max_i b_i\}$ and $|Z|$ is the cardinality of Z. Notice that the payoff of a bidder does not depend on the type of other bidders.

Formally, we define a Bayesian game as

$$\Gamma = \{N, ((T_i, \mathcal{T}_i, \mu_i), (B_i, \mathcal{B}_i), \pi_i)_{i \in N}\}$$

N is the set of bidders, $(T_i, \mathcal{T}_i, \mu_i)$ is a probability space, $(B_i, \mathcal{B}_i)$ is a measurable space and π_i is the payoff function. We assume that the measure μ_i is absolutely continuous with respect to the Lebesgue measure for every $i \in N$. The probability measure μ_i is generated from the distribution function F_i and is defined as $\mu_i(c, d] := F_i(d) - F(c)$ for every $c, d \in \Re_+$.

We define the behavioral and mixed strategy of the bidders. The function $\psi_i : T_i \times \mathcal{B}_i \to \Re$ is a **behavioral strategy** of bidder i if
(A) $\psi_i(., A_i) : T_i \to \Re$ is $\mathcal{T}_i$-measurable function for every $A_i \in \mathcal{B}_i$; and
(B) $\psi_i(t_i, .) : \mathcal{T}_i \to \Re$ is a probability measure for every $t_i \in T_i$.

The behavioral strategy is a transition function or a regular conditional distribution. We interpret $\psi_i(t_i, A_i)$ as the probability that $b_i \in A_i$ given his type $t_i \in T_i$.

The strategy ψ_i is **pure** if for every $t_i \in T_i$,
(A) there exists a singleton set S_i such that $\psi_i(t_i, S_i) = 1$; and
(B) $\psi_i(t_i, A_i) = 0$ for every $A_i \in \mathcal{T}_i$ such that $A_i \neq S_i$.

With abuse of notation, whenever $\psi_i(t_i, \{b_i\}) = 1$, we write $\psi_i(t_i) = b_i$.

We now define the mixed strategy of the bidders. The mixed strategy will be induced by the behavioral strategy. Let the space of behavioral strategy ψ_i be $\Delta(B_i)$. Consider a measurable rectangle $P_i \times Q_i \in \mathcal{T}_i \otimes \mathcal{B}_i$ such that $P_i \in \mathcal{T}_i$ and $Q_i \in \mathcal{B}_i$. Let the space of all measurable rectangles be $\mathcal{E}_i$. The pair $(T_i \times B_i, \mathcal{T}_i \otimes \mathcal{B}_i)$ is a product measurable space. We define a product measure $\varphi_i : \mathcal{T}_i \otimes \mathcal{B}_i \to \Re$ as

$$\varphi_i(P_i \times Q_i) := \int_{P_i} \mu_i(\mathrm{d}t_i)\psi_i(t_i, Q_i)$$

for every $P_i \times Q_i \in \mathcal{T}_i \otimes \mathcal{B}_i$. Note $P_i \subseteq T_i$ and $Q_i \subseteq B_i$. We interpret $\varphi_i(P_i \times Q_i)$ as the probability that $t_i \in P_i$ and $b_i \in Q_i$.

The **mixed strategy** $\beta_i : \mathcal{B}_i \to \Re$ is the marginal of φ_i on T_i, i.e., $\beta_i(Q_i) := \varphi_i(T_i, Q_i)$ for every $Q_i \in \mathcal{B}_i$. We interpret $\beta_i(Q_i)$ as the probability that $b_i \in Q_i$. Consequently, the triplet $(B_i, \mathcal{B}_i, \beta_i)$ is a measure space. On the other hand, the marginal of φ_i on B_i is $\mu_i(P_i) = \varphi_i(P_i, B_i)$.

We now define the expected utility of bidders. We shall define the expected utility function conditional on their own type. The *interim* expected utility $U_i : \Delta(B_i) \times \Delta(B_j) \times T_i \to \Re$ for bidder $i \in N$ is

$$U_i(\psi_i, \psi_j, t_i) = \int_{B_i \times T_j \times B_j} \psi_i(t_i, \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j) \pi_i(t_i, b_i, b_j)$$

for every $t_i \in T_i$.

We now define a Bayesian equilibrium in behavioral strategies.

Definition 1. *A profile of functions (ψ_i, ψ_j) is a **Bayesian equilibrium** if for every $i \in N$, $t_i \in T_i$ and $\psi_i' \in \Delta(B_i)$,*

$$U_i(\psi_i, \psi_j, t_i) \geq U_i(\psi_i', \psi_j, t_i).$$

Consider a probability bidding space $(B_i, \mathcal{B}_i, \beta_i)$. Let $g_i : B_i \to \Re$ be a $\mathcal{B}_i$-measurable random variable with respect to the probability measure β_i. We interpret g_i as the bid made by bidder i. The co-domain of the function g_i represents the set of "effective" bids when i follows a mixed strategy β_i.

Let $\{(B_i, \mathcal{B}_i, \psi_i(t_i, .)) | t_i \in T_i\}$ be a family of probability bidding spaces. For a given type t_i, consider a probability bidding space $(B_i, \mathcal{B}_i, \psi_i(t_i, .))$. Let $h_i(., t_i) : B_i \to \Re$ be a $\mathcal{B}_i$-measurable random variable with respect to the probability measure $\psi_i(., t_i)$. Let $\{h_i(., t_i) | t_i \in T_i\}$ be a family of $\mathcal{B}_i$-measurable functions. We interpret $h_i(., t_i)$ as the bid made by bidder i. The co-domain of the function $g_i(t_i)$ represents the set of bids that bidder i bids with positive probability when the behavioral strategy $\psi_i(t_i, .)$ is implemented.

To show that equilibrium is in pure strategies, we construct a correspondence indicating the set of all the bids chosen by the player for a given type. We then segregate this correspondence into two functions indicating the infimum and supremum of the set of all the bids chosen. The underlying idea is to show that, for a given type, there exists a unique bid chosen by the player in equilibrium, i.e., the correspondence is indeed a function, or equivalently, both the functions are essentially the same.

Suppose (ψ_i, ψ_j) is a Bayesian equilibrium. Let a correspondence $\Sigma_i : T_i \rightrightarrows B_i$ be defined as

$$\Sigma_i(t_i) = \{b_i \in B_i | U_i(\psi_i, \psi_j, t_i) = U_i(b_i, \psi_j, t_i)\}$$

Let the functions $\Lambda_i : T_i \to B_i$ and $\Omega_i : T_i \to B_i$ be defined as

$$\Lambda_i(t_i) = \inf \Sigma_i(t_i), \text{ and}$$
$$\Omega_i(t_i) = \sup \Sigma_i(t_i)$$

for every $i \in N$. By the definition of a Bayesian equilibrium, $U_i(\psi_i, \psi_j, t_i)$ is the maximum expected payoff generated by bidder i given that bidder j follows ψ_j. Given a type t_i, $\Sigma_i(t_i)$ is the set of all those bids which bidder i bids with positive probability. Furthermore, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are the infimum and supremum of all those bids respectively. Thus, by the definition of a random strategy equilibrium, the sets $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are non-empty for every $t_i \in T_i$ and for every $i \in N$.

In order to show that bidding strategies are measurable functions (pure strategies), we need to show that the correspondence Σ_i is indeed a function, or equivalently, the functions Λ_i and Ω_i are essentially the same. In Appendix A, we study important properties of Λ_i and Ω_i. Here, we discuss those properties very briefly. The following are the properties:
(A) $\Lambda_i(0) = \Omega_i(0) = 0$ and $\Lambda_i(a_i) = \Omega_i(a_i) = \bar{b}$ for some $\bar{b} \in \Re_{++}$.
(B) Λ_i and Ω_i are strictly increasing in types.
(C) Λ_i and Ω_i are continuous functions.

From (A) and (C), it follows that Λ_i and Ω_i are equivalent. Thus, equilibrium bidding strategies are measurable functions, strictly increasing and continuous.

It can be readily seen that

$$\psi_i(0) = 0 \quad \text{and} \quad \psi_i(t_i) = \bar{b} \quad \text{for some} \quad \bar{b} \in \Re_{++} \tag{1}$$

After establishing the measurability, monotonicity and continuity properties of the equilibrium, we are now ready to characterize all the Bayesian equilibria by deriving the differential equations. Often, it is more convenient to work with inverse bidding strategies rather than bidding strategies.

Suppose bidder i with type t_i bids b and bidder j follows his equilibrium bidding strategy ψ_j. Then, bidder i wins the auction by bidding b if and only if the type of bidder j is less than the type required by bidder j in order to bid b, i.e., $t_j < \psi_j^{-1}(b)$. Hence, bidder i wins with probability $F_j \circ \psi_j^{-1}(b)$ by bidding b. Therefore, the optimization problem of bidder i is

$$\max_b F_j \circ \psi_j^{-1}(b) u(t_i - b)$$

The first-order condition leads to the following differential equation

$$\frac{F_i \circ \psi_i^{-1}(b)}{\mathrm{D}F_i \circ \psi_i^{-1}(b)} = \frac{u(\psi_j^{-1}(b) - b)}{u'(\psi_j^{-1}(b) - b)} \tag{2}$$

We state the characterization of all Bayesian equilibria.

Theorem 1 (Characterization of equilibria)**.** *The profile of functions (ψ_i, ψ_j) is a Bayesian equilibrium if and only if for every $i \in N$,*
(A) ψ_i is pure, strictly increasing and continuous; and
(B) (ψ_i, ψ_j) solves the boundary value problem given by (1) *and* (2)*.*

Proof. Appendix B ■

The above theorem has two main insights. First, the necessary conditions characterizing the equilibria are pure strategy, strict monotonicity and continuity. Second, the differential equation given by the first-order condition of the optimization problem is both necessary and sufficient for an equilibrium.

3 Bidding behavior

In this section, we derive the necessary and sufficient conditions required to unambiguously rank the bidding strategies. To do so, we will not use the differential equations derived in section 2. Instead, we will use a different methodology as given in Kirkegaard [5]. We compare the ratio of expected utilities and the ratio of distribution functions to establish some interesting properties of the equilibrium. The expected utilities are endogenous in the model whereas the distribution functions are exogenous. Moreover, the ratio of distribution functions reflect the "relative power" of a bidder. We shall shortly see how the bidding behavior is affected by the relative power of a bidder.

To begin with, we do not restrict to any stochastic order on the probability distributions. We present the results in a more general way. First-order stochastic dominance means that the ratio of distribution functions is either more than or less than one, i.e., the distribution functions never intersect. And, conditional stochastic dominance means that the slope of the ratio of distribution functions is monotone, i.e., the slope of the ratio of distribution functions never changes its sign. We allow the distribution functions to intersect a finite number of times and the slope of the ratio of distribution functions to change its sign a finite number of times. Of course, the special cases would include the first-order stochastic dominance and conditional stochastic dominance. We are interested in answering the following questions. How many times the bidding strategies intersect when the distribution functions intersect a finite number of times, say M_2? How many times the ratio of the expected utilities will be equal to the ratio of the distribution functions when it is known that the slope of the ratio of distribution functions changes its sign a finite number of times, say M_1?

We now begin the formal analysis. Consider bidder i with type t. Suppose he follows his equilibrium bidding strategy ψ_i and the other bidder follows his equilibrium bidding strategy ψ_j. Then, the expected utility of bidder i is

$$U_i(t) = u(t - \psi_i(t))F_j \circ \zeta_j(t) \tag{3}$$

where $\zeta_j(t) = \psi_j^{-1} \circ \psi_i(t)$. In words, for a given type t of bidder i, ζ_j is the type required by bidder j to match the bid made by bidder i. In the above expression, $F_j \circ \zeta_j(t)$ is the probability that bidder i with type t will win the auction by bidding $\psi_i(t)$.

We can re-write the expected utility of bidder i in an alternate manner. Consider bidder i with type t. Suppose bidder j follows his equilibrium bidding strategy ψ_j. The optimization problem for bidder i requires to choose x in order to maximize $u(t - \psi_i(x))F_j \circ \zeta_j(x)$. Therefore, the expected utility can be re-written as

$$U_i(t) = \max_x u(t - \psi_i(x))F_j \circ \zeta_j(x) \tag{4}$$

Using Envelope Theorem in (4), we have

$$\mathrm{D}U_i(t) = u'(t - \psi_i(t))F_j \circ \zeta_j(t) \tag{5}$$

Integrating the above equation and using the fact that $U_i(0) = 0$, we get

$$U_i(t) = \int_0^t u'(x - \psi_i(x))F_j \circ \zeta_j(x)\mathrm{d}x \tag{6}$$

For this section, we assume that the type spaces are same for both the bidders and are given by $T_1 = T_2 = [0, \bar{a}]$. In the following Lemma, we capture some insights if it is given that bidder i is better off than bidder j.

Lemma 1. *If $U_i(t) > U_j(t)$ for every $t \in (0, \bar{a})$, then $F_j \circ \zeta_j(t) > F_i(t)$ and $F_i \circ \zeta_i(t) < F_j(t)$.*

Proof. Appendix B ■

The above Lemma states that for every t, if the expected utility of bidder i is more than that of bidder j, then the probability that bidder i will get a type higher than t (resp. $\zeta_i(t)$) is more than the probability that bidder j will get a type higher than $\zeta_j(t)$ (resp. t). Intuitively, this means that if bidder i is better off than bidder j, then bidder i will win more often than bidder j.

Let $\mathcal{G}_i : (0, \bar{a}] \to \Re_{++}$ be defined as

$$\mathcal{G}_i(t) = \frac{F_j(t)}{F_i(t)}$$

In words, $\mathcal{G}_i$ is the *relative power* of bidder i with respect to bidder j. We say bidder i is **strong** (resp. **weak**) if $\mathcal{G}_i$ is greater (resp. less) than one. When $\mathcal{G}_i > 1$, F_i first-order stochastically dominates F_j. This means that the relative power of bidder i is more than that of bidder j. On the contrary, when $\mathcal{G}_i$ is strictly monotonically decreasing, F_i conditional stochastically dominates F_j. This means that the relative power of bidder i decreases with increasing type. As we shall see, these two properties about the relative power play a key role in determining the bidding behavior of agents.

Let

$$\mathcal{U}_i(t) = \frac{U_i(t)}{U_j(t)}$$

In words, $\mathcal{U}_i$ is the *relative expected utility* of bidder i with respect to bidder j.

In the following Lemma, we show how bidding strategies, expected utilities and distribution functions are related.

Lemma 2. *The following statements are equivalent:*
(A) $\psi_i(t) > (=,<)\psi_j(t)$ *for every* $t \in (0,\bar{a})$.
(B) $\mathcal{U}_i(t) > (=,<)\mathcal{G}_i(t)$ *for every* $t \in (0,\bar{a})$.
(C) $\mathrm{D}\mathcal{U}_i(t) > (=,<)0$ *for every* $t \in (0,\bar{a})$.

Proof. Appendix B ■

Bidder i bids more aggressively than bidder j if and only if his relative expected utility is more than his relative power.

Let M_1 be the number of stationary points of $\mathcal{G}_i$. We interpret M_1 as the number of times the slope of $\mathcal{G}_i$ changes its sign. We assume that $M_1 < \infty$. As a special case, $M_1 = 0$ implies that the relative power of a bidder is either increasing or decreasing everywhere, i.e., the distribution function of one bidder is conditional stochastically dominant to that of the other bidder. We now partition the type space into $M_1 + 1$ intervals such that on each interval, the function $\mathcal{G}_i$ is either increasing or decreasing. Formally, the type space can be partitioned as

$$(0,\bar{a}] = \bigcup_{k=0}^{M_1} (r_k, r_{k+1}]$$

such that $r_0 = 0 < r_1 < ... < r_{M_1} < r_{M_1+1} = \bar{a}$.

Let M_2 be the number of times $\mathcal{G}_i(t) = 1$. We interpret $M_2 + 1$ as the number of times F_i is less than or greater than F_j over the whole type space. We assume that $M_2 < \infty$. As a special case, $M_2 = 0$ implies that the distribution function of one bidder first-order stochastically dominates that of the other bidder. We now partition the type space into $M_2 + 1$ intervals such that on each interval, the function $\mathcal{G}_i$ is either greater than one or less than one. Formally, the type space can be partitioned as

$$(0,\bar{a}] = \bigcup_{k=0}^{M_2} (s_k, s_{k+1}]$$

such that $s_0 = 0 < s_1 < ... < s_{M_2} < s_{M_2+1} = \bar{a}$.

Without loss of generality, assume $G_i(t) > 1$ and $\mathrm{D}\mathcal{G}_i(t) < 0$ for every $t \in (s_{M_2}, \bar{a}]$. We now analyze the behavior of the relative expected utility function when the number of times G_i equals one is finite. In the following theorem, we provide an upper bound on the number of times the expected utility of bidder i equals the expected utility of bidder j.

Theorem 2. $\mathcal{U}_i(t) = 1$ *at most* M_2 *times on* $(0,\bar{a})$.

Proof. We show that $\mathcal{U}_i(t) = 1$ at most once on each (s_k, s_{k+1}). Consider the first partition $(0, s_1)$. Suppose $\mathcal{G}_i(t) > 1$ for every $t \in (0, s_1)$. Suppose there exists $t' \in (0, s_1)$ such that $\mathcal{U}_i(t) = 1$. Then, $\mathcal{U}_i(t') < \mathcal{G}_i(t')$. From Lemma 2, $\mathcal{U}_i'(t') < 0$. We show that $\mathcal{U}_i \neq 1$ on (t', s_1). We show by contradiction. Suppose there exists $t'' \in (t', s_1)$ such that $\mathcal{U}_i(t'') = 1$. Since $\mathcal{U}'(t') < 0$, it follows that there exists $t''' \in (t', s_1)$ such that $t''' < t''$ and $\mathcal{U}_i'(t''') = 0$. From Lemma 2, $\mathcal{U}_i(t''') = \mathcal{G}_i(t''')$, which is a contradiction, as $\mathcal{U}_i(t''') < 1$ and $\mathcal{G}_i(t''') > 1$. Hence, $\mathcal{U}_i(t) = 1$ at most once on $(0, s_1)$. Similarly, $\mathcal{U}_i(t) = 1$ at most once on each (s_k, s_{k+1}). A similar argument holds if $\mathcal{G}_i(t) < 1$ for every $t \in (0, s_1)$. Therefore, $\mathcal{U}_i(t) = 1$ at most M_2 times on $(0, \bar{a})$. ■

The number of times expected utility is same for both the bidders is less than or equal to the number of times the distribution function of both the bidders intersect. A caveat of the above result is that there is no lower bound on the number of intersections of U_i and U_j. There is a possibility that $\mathcal{U}_i$ is greater than one (or less than one) on the whole type space whenever $M_2 > 0$. Nonetheless, for the special case $M_2 = 0$, some important properties are discussed in the following corollaries.

Corollary 1. *If $M_2 = 0$, then $\mathcal{U}_i(t) > 1$ for every $t \in (0, \bar{a})$.*

Proof. Follows immediately from the definition of M_2 and Theorem 2. ■

Whenever bidder i first-order stochastically dominates bidder j, bidder i generates more expected utility than bidder j.

Corollary 2. *If $M_2 = 0$, then $F_i \circ \psi_i^{-1}(b) < F_j \circ \psi_j^{-1}(b)$ for every $b \in (0, \bar{b})$.*

Proof. Follows immediately from Lemma 1 and Theorem 2. ■

Whenever bidder i first-order stochastically dominates bidder j, then bidder i produces a stronger bid distribution than bidder j, i.e., the equilibrium winning probability is more for bidder i. An important implication is that *first-order stochastic dominance is a sufficient condition to rank the bid distributions.*

Until now, we have found the upper bounds on the number of times the relative expected utility equals one. In the following theorem, we provide an upper bound on the number of times the relative expected utility equals relative power.

Theorem 3. *$\mathcal{U}_i(t) = \mathcal{G}_i(t)$ at most M_1 times on $(0, \bar{a})$.*

Proof. We show $\mathcal{U}_i(t) = \mathcal{G}_i(t)$ at most once on each (r_k, r_{k+1}). Consider the first partition $(0, r_1)$. Suppose $\mathrm{D}\mathcal{G}_i(t) > 0$ for every $t \in (0, r_1)$. Suppose there exists $t' \in (0, r_1)$ such that $\mathcal{U}_i(t') = \mathcal{G}_i(t')$. Then, by Lemma 2, $\mathrm{D}\mathcal{U}_i(t') = 0$.

Two cases are possible: either t' is a local maxima or a local minima. We consider both the cases.

Suppose first that t' is a local maxima. Then, $\mathcal{U}_i$ is decreasing in the neighborhood of t'. We show $\mathcal{U}_i \neq \mathcal{G}_i$ on (t', r_i). Suppose there exists $t'' \in (t', r_1)$ such that $\mathcal{U}'(t'') = 0$. Then, by Lemma 2, $\mathcal{U}_i(t'') = \mathcal{G}_i(t'')$ which is a contradiction, as $\mathcal{U}_i(t'') < \mathcal{U}_i(t') = \mathcal{G}_i(t') < \mathcal{G}_i(t'')$.

Now suppose, t' is a local minima. Then, $\mathcal{U}_i$ is increasing in the neighborhood of t'. By Lemma 2, $\mathcal{U}_i > \mathcal{G}_i$ in the neighborhood of t', which is a contradiction.

Hence, $\mathcal{U}_i(t) = G_i(t)$ at most once on $(0, r_1)$. Similarly, $\mathcal{U}_i(t) = G_i(t)$ at most once on each (r_k, r_{k+1}). A similar argument holds if $\mathrm{D}G_i(t) < 0$ for every $t \in (0, r_1)$. Therefore, $\mathcal{U}_i(t) = G_i(t)$ at most once on each $(r_k, r_k + 1)$. From Lemma 2, $\mathcal{U}_i(t) = G_i(t)$ at most M_1 times on $(0, \bar{a})$. ■

The number of times the relative expected utility is equal to the relative power is less than or equal to the number of times the slope of the relative distribution function changes its sign. Equivalently, the upper bound on the number of times the bidding strategies intersect is M_1. This follows directly from Lemma 2. It is worthwhile to note that whenever $M_1 > 0$ and $M_2 > 0$, there exists a $t \in (0, \bar{a})$ such that $\mathcal{U}_i(t) = \mathcal{G}_i(t)$. That is, there is a lower bound on the intersection of the relative expected utility and relative power whenever $M_1 > 0$ and $M_2 > 0$. However, there arises a possibility that the bidding strategies can be unambiguously ranked when $M_1 > 0$ and $M_2 = 0$ (first-order stochastic dominance). This means that when first-order stochastic dominance is satisfied but conditional stochastic dominance is not satisfied, then it may be possible to unambiguously compare the bidding strategies. Few observations can be made regarding the intersection of $\mathcal{U}_i$ and $\mathcal{G}_i$ whenever $M_1 = 0$ (conditional stochastic dominance). We state those observations in the following corollaries.

Corollary 3. *If $M_1 = 0$, then $\psi_j(t) > \psi_i(t)$ for every $t \in (0, \bar{a})$.*

Proof. Follows immediately from Lemma 2 and Theorem 3. ■

The first observation is that whenever bidder i conditional stochastically dominates bidder j, then bidder j bids more aggressively than bidder i. This means that *conditional stochastic dominance is a sufficient condition to unambiguously rank the bidding strategies.*

Corollary 4. *If $\psi_j(t) > \psi_i(t)$ for every $t \in (0, \bar{a})$, then $M_2 = 0$.*

Proof. Suppose $\psi_j(t) > \psi_i(t)$ for every $t \in (0, \bar{a})$. Then, by Lemma 2, $\mathrm{D}\mathcal{U}_i(t) < 0$ for every $t \in (0, \bar{a})$. Since $\mathcal{U}_i(\bar{a}) = 1$, it follows $\mathcal{G}_i(t) > \mathcal{U}_i(t) > 1$. Therefore, $M_2 = 0$. ■

The second observation is that *first-order stochastic dominance is a necessary condition to unambiguously rank the bidding strategies.*

As we have seen, Theorem 3 provides only upper bound on the number of times $\mathcal{U}_i = \mathcal{G}_i$. We shall now discuss the conditions under which we can precisely calculate the number of times $\mathcal{U}_i = \mathcal{G}_i$.

Definition 2. *$\mathcal{G}_i$ is said to have* ***damped oscillation behavior*** *if the following conditions are satisfied*
(A) $1 < \mathcal{G}_i(r_{M_1}) < \mathcal{G}_i(r_{M_1-2}) < \mathcal{G}_i(r_{M_1-4}) < \dots$
(B) $1 > \mathcal{G}_i(r_{M_1-1}) > \mathcal{G}_i(r_{M_1-3}) > \mathcal{G}_i(r_{M_1-5}) > \dots$

From the above definition, it follows that $M_1 = M_2$.

Theorem 4. *Suppose $\mathcal{G}_i$ has damped oscillation behavior. Then, $\mathcal{U}_i(t) = \mathcal{G}_i(t)$ exactly M_1 times on $(0, \bar{a})$.*

Proof. We show $\mathcal{U}_i = \mathcal{G}_i$ exactly once on each (r_k, r_{k+1}). Note that $\mathcal{G}_i$ is strictly increasing on (r_{M_1-k}, r_{M_1-k+1}) where k is odd; and $\mathcal{G}_i$ is strictly decreasing on (r_{M_1-l}, r_{M_1-l+1}) where l is even. Also note that $\mathcal{G}_i$ has a critical point at every r_k. So, $\mathcal{G}_i$ has a local maxima at r_l where l is even and a local minima at r_k where k is odd. Moreover, $\mathcal{G}_i$ equals one at every s_k. From the definition of damped oscillating behavior, $\mathcal{G}_i$ is strictly increasing at $t = s_{M_2-l}$ such that $\mathcal{G}_i(s_{M_2-l}) = 1$ where l is even; and $\mathcal{G}_i$ is strictly decreasing at $t = s_{M_2-k}$ such that $\mathcal{G}_i(s_{M_2-k}) = 1$ where k is odd.

Consider the set (r_{M_1-1}, r_{M_1}). We show that $\mathcal{U}_i = \mathcal{G}_i$ exactly once in this set. Recall that $\mathcal{G}_i(\bar{a}) = \mathcal{U}_i(\bar{a}) = 1$. From the continuity of $\mathcal{U}_i$ and the compatibility of Lemma 2, $\mathcal{U}_i = \mathcal{G}_i$ exactly once on (r_{M_1-1}, r_{M_1}). Let that point be t^*.

Now consider the set (r_{M_1-2}, r_{M_1-1}). Note that t^* is a local maxima of $\mathcal{U}_i$. From the definition of damped oscillating behavior, $\mathcal{U}_i(t^*) < \mathcal{G}_i(r_{M_1-2})$. From Lemma 2, it follows $\mathcal{U}_i = \mathcal{G}_i$ exactly once on (r_{M_1-2}, r_{M_1-1}). Using similar arguments, $\mathcal{U}_i = \mathcal{G}_i$ exactly once on each (r_k, r_{k+1}). ■

Whenever the relative power function satisfies the damped oscillation behavior, the number of times the relative expected utility equals the relative power is exactly equal to the number of critical points of the relative power function.

4 Comparative statics

We now study a comparative static problem. Suppose the distribution function of a bidder changes stochastically in a manner that the new distribution is dominant to the previous one. In other words, this bidder becomes relatively stronger. We ask the following questions: How does the behavior of bidders change due to a change in the distribution function of a bidder? Does the other bidder—whose distribution function has not changed—behave more aggressively? Does the winning probability of the bidder—whose distribution function has changed—increase? Is the bidder—whose distribution has changed—better off and the bidder—whose distribution has not changed—worse off?

Consider a bidder j with distribution function F_j. Suppose the distribution function of j changes to $\hat{F}_j$ such that $\hat{F}_j$ is conditional stochastically dominant to F_j. When distribution functions are (F_i, F_j), the bidding strategies and inverse bidding strategies are denoted by (ψ_i, ψ_j) and (θ_i, θ_j) respectively. When the distribution functions are $(F_i, \hat{F}_j)$, the bidding strategies and inverse bidding strategies are denoted by $(\hat{\psi}_j, \hat{\psi}_j)$ and $(\hat{\theta}_i, \hat{\theta}_j)$ respectively.

When distribution functions are (F_i, F_j), the characterization of inverse bidding strategies is given by

$$\begin{aligned} &\mathrm{D}\theta_j(b) = \frac{F_j \circ \theta_j(b)}{f_j \circ \theta_j(b)} \frac{u'(\theta_i(b) - b)}{u(\theta_i(b) - b)} \\ &\mathrm{D}\theta_i(b) = \frac{F_i \circ \theta_i(b)}{f_i \circ \theta_i(b)} \frac{u'(\theta_j(b) - b)}{u(\theta_j(b) - b)} \\ &\theta_i(0) = \theta_j(0) = 0 \\ &\theta_i(\bar{b}) = a_i \quad \& \quad \theta_j(\bar{b}) = a_j \quad \exists \quad \bar{b} \in \Re_{++} \end{aligned} \tag{7}$$

When distribution functions are $(F_i, \hat{F}_j)$, the the characterization of inverse bidding strategies is given by

$$\begin{aligned} &\mathrm{D}\hat{\theta}_j(b) = \frac{\hat{F}_j \circ \hat{\theta}_j(b)}{\hat{f}_j \circ \hat{\theta}_j(b)} \frac{u'(\hat{\theta}_i(b) - b)}{u(\hat{\theta}_i(b) - b)} \\ &\mathrm{D}\hat{\theta}_i(b) = \frac{F_i \circ \hat{\theta}_i(b)}{f_i \circ \hat{\theta}_i(b)} \frac{u'(\hat{\theta}_j(b) - b)}{u(\hat{\theta}_j(b) - b)} \\ &\hat{\theta}_i(0) = \hat{\theta}_j(0) = 0 \\ &\hat{\theta}_i(\hat{b}) = a_i \quad \& \quad \hat{\theta}_j(\hat{b}) = a_j \quad \exists \quad \hat{b} \in \Re_{++} \end{aligned} \tag{8}$$

We state the following result.

Theorem 5. *Suppose the profiles of measurable functions (ψ_i, ψ_j) and $(\hat{\psi}_i, \hat{\psi}_j)$ are a Bayesian equilibrium when the distribution functions are (F_i, F_j) and $(F_i, \hat{F}_j)$ respectively. Suppose $\hat{F}_j$ is conditional stochastic dominant to F_j and $F_i(0), F_j(0) > 0$. Then,*
(A) $\hat{\psi}_i(t_i) > \psi_i(t_i)$ for every $t_i \in T_i - \{0\}$,
(B) $\hat{F}_j \circ \hat{\theta}_j(b) < F_j \circ \theta_j(b)$ for every $b \in \bar{B} - \{0\}$.

Proof. Appendix B ■

A stochastic change in the probability distribution of bidder j induces bidder i to bid more aggressively. It also increases the winning probability of bidder j. To establish this result, the key idea is to show that $\bar{b} < \hat{b}$. Once we have shown that $\bar{b} < \hat{b}$, the exposition becomes relatively simpler. Since $\bar{b} < \hat{b}$, bidder i bids more aggressively than before around the neighborhood of $\bar{b}$. All that remains to show is that the two bidding strategies do not cross. But to show $\bar{b} < \hat{b}$ is not straightforward. We begin the analysis by assuming the counterfactual fact that $\theta_i \leq \hat{\theta}_i$ and $\theta_j \leq \hat{\theta}_j$, and then show that both the inequalities cannot hold together. Since $\theta_i \leq \hat{\theta}_i$ and $\theta_j \leq \hat{\theta}_j$ cannot hold simultaneously, it must be true that $\bar{b} < \hat{b}$.

When distribution functions are (F_i, F_j), we denote the equilibrium expected payoffs by $U_i(t_i, \psi_i, \psi_j)$ and $U_j(t_j, \psi_i, \psi_j)$. On the other hand, when distribution functions are $(F_i, \hat{F}_j)$, we denote the equilibrium expected payoffs by $\hat{U}_i(t_i, \hat{\psi}_i, \hat{\psi}_j)$ and $\hat{U}_j(t_j, \hat{\psi}_i, \hat{\psi}_j)$.

An immediate corollary of the above theorem is that both the bidders are worse off due to a stochastic change in the distribution function. This result is stated below.

Corollary 5. *Suppose the profiles of measurable functions (ψ_i, ψ_j) and $(\hat{\psi}_i, \hat{\psi}_j)$ are a Bayesian equilibrium when the distribution functions are (F_i, F_j) and $(F_i, \hat{F}_j)$ respectively. Suppose $\hat{F}_j$ is conditional stochastic dominant to F_j and $F_i(0), F_j(0) > 0$. Then, $\hat{U}_i(t_i, \hat{\psi}_i, \hat{\psi}_j) < U_i(t_i, \psi_i, \psi_j)$ and $\hat{U}_j(t_j, \hat{\psi}_i, \hat{\psi}_j) < U_j(t_j, \psi_i, \psi_j)$ for every $t_i \in T_i - \{0\}$ and $t_j \in T_j - \{0\}$.*

Proof. Appendix B ■

Since bidder j produces a stronger bid distribution after the distribution change, it decreases the winning probability of bidder i for every b. Hence, bidder i is worse off. On the contrary, as bidder i bids more aggressively after the distribution change, it reduces the winning probability of bidder j for every b. Thus, bidder j is worse off.

Notice that in establishing any of the above results, we made no assumption on the type of asymmetries of the bidders. We discuss two special cases here. Suppose bidder i is weak, bidder j is strong and $\hat{F}_j = F_i$. Then, the weak bidder bids more aggressively while playing against a strong bidder than playing against another weak bidder. On the contrary, suppose bidder i is strong, bidder j is weak and $\hat{F}_j = F_i$. Then, the strong bidder bids more aggressively while playing against a strong bidder than playing against another weak bidder.

5 Different types of risk aversion

In this section and henceforth, we assume that one bidder is strong (s) and the other is weak (w), i.e., $N = \{s, w\}$. We provide bidders' preferences for first-price and second-price auction under different types of absolute risk aversion. We assume that the reserve price of $r \in \Re_{++}$ is exogenously given.

We make the following assumption on the distribution functions.

Assumption 1. *F_s is conditional stochastic dominant to F_w, i.e.,*

$$\frac{f_s(x)}{F_s(x)} > \frac{f_w(x)}{F_w(x)}$$

for every $x \in \Re_+$.

The following Lemma states that the weak bidder bids more aggressively and produces a weaker bid distribution than the strong bidder.

Lemma 3. *Suppose Assumption 1 is satisfied. Then,*
(A) $\psi_w(t) > \psi_s(t)$ *for every* $t \in T_w$.
(B) $F_s \circ \theta_s(b) < F_w \circ \theta_w(b)$ *for every* $b \in (0, \bar{b})$.

Proof. Appendix B[3] ■

The Arrow-Pratt measure of absolute risk aversion is defined as $A(x) = -u''(x)/u'(x)$ for every $x \in \Re$. The utility function u has increasing absolute risk aversion if $\mathrm{D}A(x) > 0$, has constant absolute risk aversion if $\mathrm{D}A(x) = 0$ and has decreasing absolute risk aversion if $\mathrm{D}A(x) < 0$. The following properties are useful in establishing the bidders' preferences for different types of absolute risk aversion[4]:
(A) With increasing absolute risk aversion, $u(x-y) = \mathrm{E}_{\hat{y}}(u(x-\hat{y}))$ implies $u'(x-y) > \mathrm{E}_{\hat{y}}(u'(x-\hat{y}))$.
(B) With constant absolute risk aversion, $u(x-y) = \mathrm{E}_{\hat{y}}(u(x-\hat{y}))$ implies $u'(x-y) = \mathrm{E}_{\hat{y}}(u'(x-\hat{y}))$.
(C) With decreasing absolute risk aversion, $u(x-y) = \mathrm{E}_{\hat{y}}(u(x-\hat{y}))$ implies $u'(x-y) < \mathrm{E}_{\hat{y}}(u'(x-\hat{y}))$.

Consider a first-price auction. The value function of bidder i is given by

$$V_i^I(t_i) = F_j \circ \theta_j(b^*) u(t_i - b^*) \tag{9}$$

where

$$b^* \in \arg\max_b F_j \circ \theta_j(b) u(t_i - b)$$

Using Envelope Theorem, we have

$$\mathrm{D}V_i^I(t_i) = F_j \circ \theta_j(b^*) u'(t_i - b^*) \tag{10}$$

Now consider a second-price auction. It can be easily checked that, in a second-price auction, it is a weakly dominant strategy to bid your own type. Thus, the value function of bidder is

$$V_i^{II}(t_i) = F_j(t_i) \int_0^{t_i} F_j(\mathrm{d}t_j) u(t_i - t_j) \tag{11}$$

Using Envelope Theorem, we have

$$\mathrm{D}V_i^{II}(t_i) = F_j(t_i) \int_0^{t_i} F_j(\mathrm{d}t_j) u'(t_i - t_j) \tag{12}$$

Our task is to compare V_i^I and V_i^{II} under different types of absolute risk aversion and for every $i \in N$.

We state the following result.

[3]The proof of this result can also be found in Li and Riley [11].
[4]The proof of this result can be found in Lemma 1 of Maskin and Riley [12].

Theorem 6. *Suppose* (ψ_w, ψ_s) *is a Bayesian equilibrium and Assumption 1 is satisfied. Then, for every* $t > r$*, the following holds:*
(A) With increasing absolute risk aversion, $V_w^I(t) > V_w^{II}(t)$.
(B) With constant absolute risk aversion, $V_w^I(t) > V_w^{II}(t)$ *and* $V_s^{II}(t) > V_s^I(t)$.
(C) With decreasing absolute risk aversion, $V_s^{II}(t) > V_s^I(t)$.

Proof. We prove for increasing absolute risk aversion. Consider the weak bidder. Notice that $V_w^I(r) = 0 = V_w^{II}(r)$. We show $V_w^I(t) > V_w^{II}(t)$ for every $t_w > r$. It suffices to show if $V_w^I(t) = V_w^{II}(t)$, then $\mathrm{D}V_w^I(t) > \mathrm{D}V_w^{II}(t)$ for every $t > r$.

Suppose $V_w^I(t) = V_w^{II}(t)$. Then, from (10) and (12), we have

$$\mathrm{D}V_w^I(t) > \mathrm{D}V_w^{II}(t)$$

Hence, $V_w^I(t) > V_w^{II}(t)$ for every $t > r$. A similar proof holds for constant and decreasing absolute risk aversion. ■

The above result states that the weak bidder prefers first-price auction over second-price auction under increasing and constant absolute risk aversion whereas the strong bidder prefers second-price auction over first-price auction under constant and decreasing absolute risk aversion.

6 Impact of asymmetry

In this section, we study the impact of asymmetry on seller's expected revenue in a first-price auction. Does the seller prefer asymmetry over symmetry? As we shall shortly see, asymmetry makes the seller worse-off. More generally, we compare the seller's revenue for different degrees of asymmetry.

Let $\mathcal{V}$ be the class of increasing absolute risk aversion utility functions of the form

$$u(x) = x^\alpha; \quad 0 < \alpha < 1$$

Let $v : \Re_+ \to \Re$ be defined as $v = u/u'$. Notice that $v(0) = 0$, $v' > 0$ and $v'' = 0$. For this section, we assume that $a_w = a_s \equiv \bar{a}$.

We define how the distribution functions of symmetric and asymmetric bidders are related. Let F denote the distribution function of the symmetric bidders. The distribution functions F, F_s and F_w are related as

$$F(t) = \sqrt{F_w(t)F_s(t)}$$

for every $t \in [0, \bar{a}]$. Thus, we have assumed that the symmetric distribution function is the geometric average of the asymmetric distribution functions. Notice that $F(0) = 0$, $F(\bar{a}) = 1$ and $F_s(t) < F(t) < F_w(t)$ for every $t \in (0, \bar{a})$.

We now define a special family of distribution functions. Let $\mathcal{F}_1$ be the family of distribution functions of the form

$$F_i(t) = \frac{t^{c_i}}{\bar{a}}$$

where $0 < c_i \leq 1$.

Let $G(b) = F_s \circ \theta_s(b) F_w \circ \theta_w(b)$. Then, the seller's *ex-ante* expected revenue is

$$R(F_w, F_s) = \int_B G(\mathrm{d}b) b$$

Lemma 4. *Suppose (ψ_w, ψ_s) is a Bayesian equilibrium and Assumption 1 is satisfied. Then, $\psi_w(t) > \psi_s(t)$ for every $t \in T_i - \{0, \bar{a}\}$.*

Proof. From Assumption 1, $f_w/F_w < f_s/F_s$. Notice that $\zeta_s(t) = \theta_s \circ \psi_w(t)$. If $\zeta_s(t) = t$, then $\zeta_s'(t) < 1$. As $\zeta_s(\bar{a}) = \bar{a}$ and $\zeta_s(t) = t$ implies $\zeta_s(t) < t$, it follows $\zeta_s(t) > t$. Therefore, $\psi_w(t) > \psi_s(t)$ for every $t \in T_i - \{0, \bar{a}\}$. ∎

We present the main result of this section. In this result, we compare the seller's revenue for different degrees of asymmetry.

Theorem 7. *Let $F_i, \tilde{F}_i \in \mathcal{F}_1$ and $u \in \mathcal{V}$ such that $c_s + c_w = \tilde{c}_s + \tilde{c}_w$. Suppose (ψ_w, ψ_s) and $(\tilde{\psi}_w, \tilde{\psi}_s)$ are a Bayesian equilibrium when the distribution functions are (F_w, F_s) and $(\tilde{F}_w, \tilde{F}_s)$ respectively. Suppose $\tilde{c}_s > c_s \geq c_w$. Then,*

$$R(F_w, F_s) > R(\tilde{F}_w, \tilde{F}_s).$$

Proof. Appendix B ∎

The above result states that, as the degree of asymmetry increases, the seller's expected revenue decreases. The idea of the proof is as follows. Let $\tilde{G}(b) = \tilde{F}_s \circ \tilde{\theta}_s(b) \tilde{F}_w \circ \tilde{\theta}_w(b)$. To show $R(F_w, F_s) > R(\tilde{F}_w, \tilde{F}_s)$, it suffices to show $G(b) < \tilde{G}(b)$ for every $b \in \Re_{++}$. Since $G(0) = \tilde{G}(0) = 0$, we first show that $G < \tilde{G}$ around some neighborhood of 0, and then show that the two functions do not intersect. A special case of this result is that the seller's expected revenue is more in the symmetric framework as compared to the asymmetric framework. This result is stated in the following corollary.

Corollary 6. *Let $F \in \mathcal{F}_1$ and $u \in \mathcal{V}$. Then,*

$$R(F, F) > R(\tilde{F}_w, \tilde{F}_s).$$

Proof. Set $c_s = c_w = (\tilde{c}_s + \tilde{c}_w)/2$. ∎

The above result tells that the seller's expected revenue is more in the symmetric framework than in the asymmetric framework, i.e., the seller prefers symmetry over asymmetry.

7 Revenues

In this section, we analytically compute a Bayesian equilibrium in linear bidding strategies for a special family of increasing absolute risk averse utility functions and a special family of "power" distribution functions. We then compare the seller's revenue for a first-price and a second-price auction.

Let $\mathcal{F}_2$ be the family of distribution functions of the form

$$F_i(t) = \left(\frac{t}{a_i}\right)^{\tau_i+1}$$

The family of distributions in $\mathcal{F}_2$ are known as *power distributions*. The parameter τ_i captures the *power* of bidder i. A higher value of τ_i represents that bidder i is more likely to get a higher type. Consider $F_w, F_s \in \mathcal{F}_2$ and $u \in \mathcal{V}$ such that $\tau_s > \tau_w$ and $\tau_s, \tau_w \in \Re$. The bidding strategies are

$$\begin{aligned} \psi_w(t) &= \frac{\tau_s + 1}{\tau_s + 1 + \alpha} t \\ \psi_s(t) &= \frac{\tau_w + 1}{\tau_w + 1 + \alpha} t \end{aligned} \tag{13}$$

The corresponding inverse bidding strategies are

$$\begin{aligned} \theta_w(b) &= \frac{\tau_s + 1 + \alpha}{\tau_s + 1} b \\ \theta_s(b) &= \frac{\tau_w + 1 + \alpha}{\tau_w + 1} b \end{aligned} \tag{14}$$

We observe few properties of the bidding function.

Remark 1. *As $\alpha \downarrow 0$, $\psi_i(t)$ approaches t, i.e., as the degree of risk aversion increases, the bidder bids more aggressively.*

Remark 2. *The weak bidder bids more aggressively than the strong bidder, i.e. $\psi_w(t) > \psi_s(t)$ for every $t \in [0, a_w]$.*

Remark 3. *Bidder i's strategy depends on the power of bidder j and not on his own power, i.e., a change in the power coefficient of bidder i does not alter his bidding strategy.*

Remark 4. *Suppose bidder i with type t wins the auction by bidding $b := \psi_i(t_i)$. Then,*

$$E(\textit{second highest type}|\textit{bidder } i \textit{ has highest type}) = \frac{\tau_j + 1}{\tau_j + 2} t$$

Notice that, for a given type, the equilibrium winning bid is more than the expected value of the second highest type.

Recall that $\zeta_j(t) = \psi_j^{-1} \circ \psi_i(t)$ is the type required by bidder j in order to bid the same value as bidder i bids with type t. Thus, $\zeta_w(t) = kt$ where

$$k = \frac{(\tau_s + 1 + \alpha)(\tau_w + 1)}{(\tau_w + 1 + \alpha)(\tau_s + 1)}$$

Similarly, $\zeta_s(t) = t/k$. For analytical solution, we assume $a_w = ka_s$.

The equilibrium probabilities of winning for the weak bidder and the strong bidder respectively are

$$F_s \circ \zeta_s(t) = \left(\frac{t}{a_w}\right)^{\tau_s+1}$$

and

$$F_w \circ \zeta_w(t) = \left(\frac{t}{a_s}\right)^{\tau_w+1}$$

Since bid-your-own-type is a dominant strategy in a second-price auction, the equilibrium winning probabilities for the weak and the strong bidder respectively are

$$F_s(t) = \left(\frac{t}{a_s}\right)^{\tau_s+1}$$

and

$$F_w(t) = \left(\frac{t}{a_w}\right)^{\tau_w+1}$$

The maximum bid by any bidder is

$$\bar{b} = \frac{\tau_w + 1}{\tau_w + 1 + \alpha} a_s = \frac{\tau_s + 1}{\tau_s + 1 + \alpha} a_w$$

We now calculate the *ex-ante* expected revenue of the seller from the first-price and the second-price auction. In a first-price auction, the *ex-ante* expected revenue of the seller generated from the weak bidder is

$$\begin{aligned} R_w^I &= \int_0^{\bar{b}} F_w \circ \theta_w(\mathrm{d}b) b F_s \circ \theta_s(b) \\ &= \frac{(\tau_w + 1)(\tau_s + 1)}{(\tau_w + \tau_s + 3)(\tau_s + 1 + \alpha)} a_w \end{aligned}$$

Similarly, the *ex-ante* expected revenue of the seller generated from the strong bidder is

$$\begin{aligned} R_s^I &= \int_0^{\bar{b}} F_s \circ \theta_s(\mathrm{d}b) b F_w \circ \theta_w(b) \\ &= \frac{(\tau_w + 1)(\tau_s + 1)}{(\tau_w + \tau_s + 3)(\tau_w + 1 + \alpha)} a_s \end{aligned}$$

Hence, the *ex-ante* expected revenue of the seller is

$$\begin{aligned} R^I &= R_w^I + R_s^I \\ &= \frac{(\tau_w + 1)(\tau_w + \tau_s + 2)}{(\tau_w + \tau_s + 3)(\tau_w + 1 + \alpha)} a_s \end{aligned} \tag{15}$$

In a second-price auction, the *ex-ante* expected revenue of the seller is

$$\begin{aligned} R^{II} &= \int_0^{a_w} (1 - F_w(t))(1 - F_s(t))\mathrm{d}t \\ &= \frac{\tau_w + 1}{\tau_w + 2} a_w - \frac{\tau_w + 1}{(\tau_s + 2)(\tau_w + \tau_s + 3)} k^{\tau_s + 1} a_w \end{aligned} \tag{16}$$

Since the revenue expressions would be quite useful, we record them in the following Lemma.

Lemma 5. *Suppose $F_s, F_w \in \mathcal{F}_2$ and $u \in \mathcal{V}$. Then, the seller's ex-ante expected revenues generated from the first-price and second-price auctions are given by* (15) *and* (16) *respectively.*

In the following result, we compare the seller's revenue.

Theorem 8. *Suppose $F_s, F_w \in \mathcal{F}_2$, $u \in \mathcal{V}$ and bidding strategies are linear. Then, the seller's ex-ante expected revenue is more in first-price auction than in second-price auction.*

Proof. Appendix B ■

Example 1. *Suppose $u(t) = \sqrt{t}, a_w = 5, a_s = 6, \tau_w = -0.5$ and $\tau_s = -0.25$. Then, the probability distributions are $F_s(t) = (t/6)^{3/4}$ and $F_w(t) = (t/5)^{1/2}$. Under first-price auctions, the equilibrium bidding strategies are $\psi_w(t) = (3/5)t$ and $\psi_s(t) = (1/2)t$ with equilibrium inverse bidding strategies $\theta_w(b) = (5/3)b$ and $\theta_s(b) = 2b$. The equilibrium winning probabilities for the weak and strong bidder are $(t/5)^{3/4}$ and $(t/6)^{1/2}$ respectively. On the other hand, under second-price auctions, the equilibrium winning probabilities for the weak and strong bidder are $(t/6)^{3/4}$ and $(t/5)^{1/2}$ respectively. The maximum bid by any bidder is $\bar{b} = 12$. The ex-ante expected revenue of the seller under first-price auction and second-price auction is $R^I = 1.67$ and $R^{II} = 1.39$ respectively.*

8 Conclusion

In this paper, we have shown that the necessary conditions characterizing the equilibria are strict monotonicity, continuity and pure strategy. The system of differential equations are both necessary and sufficient condition for an equilibrium. We have compared the bidding strategies by not restricting only to stochastic orders. Of course, stochastic orders on distribution functions are required to unambiguously rank the bidding strategies and bidding distributions. The special cases of our result convey that first-order stochastic dominance is a necessary condition and conditional stochastic dominance is a sufficient condition to unambiguously rank the bidding strategies.

We have compared the bidders' preferences for first-price and second-price auction under different types of risk aversion. We have shown that the strong bidder prefers a second-price auction when there is constant or decreasing absolute risk aversion, and the weak bidder prefers a first-price auction when there is increasing or constant absolute risk aversion. We have also studied the impact of asymmetry on seller's revenue. We have shown that, for a special family of utility functions and distribution functions, asymmetry makes the seller worse-off. Finally, by restricting our attention to linear bidding strategies, we have shown that, for a special family of utility functions and distribution functions, a first-price auction is revenue superior to a second-price auction.

A Appendix: Properties of Λ_i and Ω_i

Lemma A.1. *Suppose the profile of transition functions* (ψ_i, ψ_j) *is a Bayesian equilibrium. Then,* $U_i(\psi_i, \psi_j, t_i) > 0$ *and* $\Pr(i \text{ wins}|t_i) > 0$ *for every* $t_i \in T_i - \{0\}$ *and* $i \in N$*.*

Proof. We show by contradiction. Suppose, for some $i \in N$, there exists $t_i > 0$ such that $U_i(\psi_i, \psi_j, t_i) = 0$. Then, for every $b_i \in \Sigma_i(t_i)$, we have $\Pr(i \text{ wins}|t_i, b_i) = 0$. Thus, for every $b_j \in \Sigma_j(t_j)$, we have $\Pr(b_j > t_i) = 1$. Since $\Pr(t_j < t_i) > 0$, we have $U_j(\psi_i, \psi_j, t_j) < 0$, which is not possible. Hence, $U_i(\psi_i, \psi_j, t_i) > 0$ for every $t_i > 0$.

Since $U_i(\psi_i, \psi_j, t_i) > 0$, it follows $\Pr(i \text{ wins}|t_i) > 0$ for every $t_i > 0$. ■

Lemma A.2. *Suppose the profile of transition functions* (ψ_i, ψ_j) *is a Bayesian equilibrium. Then,* $\Lambda_i(0) = \Omega_i(0) = 0$ *and there exists* $\bar{b}$ *such that* $\Lambda_i(a_i) = \Omega_i(a_i) = \bar{b}$ *for every* $i \in N$*.*

Proof. Let $\underline{g}_i = \inf\{g_i\}$ and $\underline{g} = \max\{\underline{g}_i, \underline{g}_j\}$. We show $\underline{g}_i = 0$ for every $i \in N$. We show by contradiction. Suppose $\underline{g} > 0$. Without loss of generality, assume $\underline{g} = \underline{g}_i$. Then, for $t_i \in (0, \underline{g})$, $\Pr(i \text{ wins}|t_i) > 0$ and $U_i(\psi_i, \psi_j, t_i) < 0$ which contradicts Lemma A.1. Hence, $\underline{g}_i = 0$ for every $i \in N$. Since $\underline{g}_i = 0$, we have $\Lambda_i(0) = \Omega_i(0) = 0$ for every $i \in N$.

Let $\bar{g}_i = \sup\{g_i\}$ and $\bar{g} = \max\{\bar{g}_i, \bar{g}_j\}$. We show that $\bar{g}_i = \bar{g}_j \equiv \bar{b}$. We show by contradiction. Suppose $\bar{g}_i > \bar{g}_j$. Then, $\Pr(i \text{ wins}|t_i = a_i, b_i = \bar{g}) = 1$. This implies there exists $\epsilon > 0$ such that $U_i(\bar{g} - \epsilon, \psi_j, a_i) > U_i(\bar{g}, \psi_j, a_i)$, which is a contradiction. Hence, $\bar{g}_i = \bar{g}_j \equiv \bar{b}$ for every $i \in N$. ■

Lemma A.3. *Suppose the profile of transition functions* (ψ_i, ψ_j) *is a Bayesian equilibrium. Then,* $\Pr(i \text{ wins}|t_i)$ *is non-decreasing in* t_i*.*

Proof. Consider any bidder i and $t_i, t_i' \in T_i$ such that $t_i' > t_i$. Then, by the definition of Bayesian equilibrium, we have

$$U_i(\psi_i(t_i, .), \psi_j, t_i) = \int \psi_i(t_i, \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i, b_i, b_j) \geq$$

$$U_i(\psi_i(t_i', .), \psi_j, t_i) = \int \psi_i(t_i', \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i, b_i, b_j)$$

Since $u(.)$ is strictly increasing and strictly concave, we have

$$u(t_i - b_i) > u(t_i' - b_i) - (t_i' - t_i)$$

Using this in the definition of π_i, we have

$$\pi_i(t_i, b_i, b_j) \geq \pi_i(t_i', b_i, b_j) - I_i(t_i' - t_i)$$

where

$$I_i = \begin{cases} 1/|Z| & \text{if} \quad i \in Z \\ 0 & \text{if} \quad i \notin Z \end{cases}$$

Multiplying the above expression with the probability that bidder i wins conditional on the type t_i', we have

$$\begin{aligned}\int \psi_i(t_i', \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i, b_i, b_j) \geq \int \psi_i(t_i', \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i', b_i, b_j) \\ - I_i(t_i' - t_i) \int \psi_i(t_i', \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\end{aligned}$$

Since $U_i(\psi_i(t_i, .), \psi_j, t_i) \geq U_i(\psi_i(t_i', .), \psi_j, t_i)$, it follows that

$$\begin{aligned}\int \psi_i(t_i, \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i, b_i, b_j) \geq \int \psi_i(t_i', \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\pi_i(t_i', b_i, b_j) \\ - I_i(t_i' - t_i) \int \psi_i(t_i', \mathrm{d}b_i) \otimes \varphi_j(\mathrm{d}t_j, \mathrm{d}b_j)\end{aligned}$$

Thus,

$$U_i(\psi_i(t_i, .), \psi_j, t_i) \geq U_i(\psi_i(t_i', .), \psi_j, t_i') - (t_i' - t_i)\Pr(i \text{ wins}|t_i')$$

Interchanging the roles of t_i and t_i', we get

$$U_i(\psi_i(t_i', .), \psi_j, t_i') \geq U_i(\psi_i(t_i, .), \psi_j, t_i) - (t_i - t_i')\Pr(i \text{ wins}|t_i)$$

The above two inequalities imply

$$\begin{aligned}(t_i' - t_i)\Pr(i \text{ wins}|t_i') &\geq U_i(\psi_i(t_i', .), \psi_j, t_i') - U_i(\psi_i(t_i, .), \psi_j, t_i) \\ &\geq -(t_i - t_i')\Pr(i \text{ wins}|t_i)\end{aligned}$$

Since $t_i' > t_i$, we have

$$\Pr(i \text{ wins}|t_i') \geq \Pr(i \text{ wins}|t_i)$$

Hence, $\Pr(i \text{ wins}|t_i)$ is non-decreasing in t_i. ■

Lemma A.4. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are non-decreasing in t_i for every $t_i \in T_i$ and $i \in N$.*

Proof. We first show $\Lambda_i(t_i') \geq \Omega_i(t_i)$ for every $t_i, t_i' \in T_i$ such that $t_i' > t_i$. We show by contradiction. Suppose there exists some bidder i such that $\Omega_i(t_i) > \Lambda_i(t_i')$. Consider two sequences $(b_n)_{n=1}^{\infty}$ and $(b_n')_{n=1}^{\infty}$ such that $b_n \downarrow \Omega_i(t_i)$ and $b_n' \uparrow \Lambda_i(t_i')$ for every $n \in \mathcal{N}$. Notice that $b_n > b_n'$ for every $n \in \mathcal{N}$. By the definition of Bayesian equilibrium, we have

$$U_i(b_n, \psi_j, t_i) = \Pr(i \text{ wins}|b_n)u(t_i - b_n) \geq \Pr(i \text{ wins}|b_n')u(t_i - b_n') = U_i(b_n', \psi_j, t_i)$$

Similarly,

$$U_i(b_n', \psi_j, t_i') = \Pr(i \text{ wins}|b_n')u(t_i' - b_n') \geq \Pr(i \text{ wins}|b_n)u(t_i' - b_n) = U_i(b_n, \psi_j, t_i')$$

Adding the above two inequalities, we get

$$\Pr(i \text{ wins}|b_n')\{u(t_i' - b_n') - u(t_i - b_n')\} \geq \Pr(i \text{ wins}|b_n)\{u(t_i' - b_n) - u(t_i - b_n)\}$$

Since $u(t_i' - b_n') - u(t_i - b_n') < u(t_i' - b_n) - u(t_i - b_n)$, we have

$$\Pr(i \text{ wins}|b_n') > \Pr(i \text{ wins}|b_n)$$

which is a contradiction, from Lemma A.3, as $b_n > b_n'$.

We now show that $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are non-decreasing in t_i. Since $\Lambda_i(t_i') \geq \Omega_i(t_i) \geq \Lambda_i(t_i)$, we have $\Lambda_i(t_i') \geq \Lambda_i(t_i)$. Since $\Omega_i(t_i') \geq \Lambda_i(t_i') \geq \Omega_i(t_i)$, we have $\Omega_i(t_i') \geq \Omega_i(t_i)$. ■

Lemma A.5. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i) > 0$ and $\Omega_i(t_i) > 0$ for every $t_i \in T_i - \{0\}$ and $i \in N$.*

Proof. We show by contradiction. Suppose there exists $i \in N$ and $t_i > 0$ such that $\Lambda_i(t_i) = 0$. Consider a sequence $(b_n)_{n=1}^{\infty}$ such that $b_n \uparrow \Lambda_i(t_i)$ for every $n \in \mathcal{N}$. Then,

$$U_i(b_n, \psi_j, t_i) = u(t_i - b_n) \Pr(i \text{ wins}|b_n)$$

for every $n \in \mathcal{N}$.

As $n \to \infty$, from Lemma A.1, we have

$$\lim_{n\to\infty} U_i(b_n, \psi_j, t_i) > 0$$

Thus,

$$\lim_{n\to\infty} \Pr(i \text{ wins}|t_i, b_n) > 0$$

Since $\Lambda_i(t_i) = 0$, from Lemma A.4, we have $\Lambda_i(s_i) = 0$ for every $s_i \in (0, t_i)$. From Lemma A.4, we know $\Omega_i(t_i) \geq \Lambda_i(t_i) \geq \Omega_i(s_i)$ and thus, $\Omega_i(t_i) = \Omega_i(s_i) = 0$. So, $\Sigma_i(t_i) = \Sigma_i(s_i) = 0$.

Since $\lim_{n\to\infty} \Pr(i \text{ wins}|t_i, b_n) > 0$, it follows that there is a strictly positive probability of a tie. Then, there exists $\epsilon > 0$ such that

$$U_i(\epsilon, \psi_j, s_i) > U_i(0, \psi_j, s_i)$$

for every $s_i \in (0, t_i)$, which is a contradiction. Hence, $\Lambda_i(t_i) > 0$ and $\Omega_i(t_i) > 0$ for every $t_i \in T_i - \{0\}$ and every $i \in N$. ■

Lemma A.6. *Suppose the profile of transition functions* (ψ_i, ψ_j) *is a Bayesian equilibrium. Then,* $\varphi_i(T_i, \{b\}) = 0$ *and* $\Pr(i \text{ wins}|b_i)$ *is continuous in every* $b_i \in B_i$ *and* $i \in N$.

Proof. We show the result in three steps. In step 1, we show that if $\varphi_i(T_i, \{b\}) > 0$ for every $b > 0$, then, for every $\epsilon > 0$, $\varphi_j(T_j, (b-\epsilon, b]) > 0$. In step 2, we show that if $\varphi_i(T_i, (b-\epsilon, b]) > 0$ for every $b > 0$ and every $\epsilon > 0$, then $\varphi_j(T_j, (b-\epsilon, b]) = 0$. In step 3, we show $\varphi_i(T_i, \{b\}) = 0$ for every $b > 0$.

Step 1: Suppose $\varphi_i(T_i, \{b\}) > 0$ for every $b > 0$. Suppose there exists $\epsilon > 0$ such that $\varphi_j(T_j, (b-\epsilon, b]) = 0$. We show a contradiction. Since $\varphi_i(T_i, \{b\}) > 0$, we have $U_i(b, \psi_j, t_i) > 0$. Since $\varphi_j(T_j, (b-\epsilon, b]) = 0$, bidder i can increase his utility by reducing his bid slightly, which is a contradiction. Hence, if $\varphi_i(T_i, \{b\}) > 0$, then, for every $\epsilon > 0$, $\varphi_j(T_j, (b-\epsilon, b]) > 0$.

Step 2: We show by contradiction. Suppose $\varphi_i(T_i, (b-\epsilon, b]) > 0$ and $\varphi_j(T_j, (b-\epsilon, b]) > 0$ for every $b > 0$ and for every $\epsilon > 0$. Consider a set $C_\epsilon \in \mathcal{B}_i(T_i)$ such that $\Pr(h_i(., t_i) \in (b-\epsilon, b]) > 0$ for every $\epsilon > 0$ and $t_i \in C_\epsilon$. Then, $\epsilon' > \epsilon$ implies $C_\epsilon \subseteq C_{\epsilon'}$.

We show there exists $\hat{\epsilon} > 0$ such that $\hat{\epsilon} < b < t_i - b$ for every $t_i \in C_{\hat{\epsilon}}$. We show by contradiction. Suppose, for every C_n, there exists $t_{in} \in C_n$ such that $t_{in} \leq n + b$ for every $n \in \mathcal{N}$. It can be verified that $t_{in} > n - b$ for every $t_{in} \in C_n$ and $n \in \mathcal{N}$. Thus,

$$U_i(\psi_i, \psi_j, t_{in}) \leq u(2n)$$

As $n \to 0$, we have $t_{in} \to b$ and thus,

$$U_i(\psi_i, \psi_j | t_{in}) \leq u(0) = 0$$

Since $U_i(\psi_i, \psi_j, b) \not< 0$, we have $U_i(\psi_i, \psi_j, b) = 0$, which is contradiction from Lemma A.1. Hence, such $\hat{\epsilon}$ exists.

Consider $\epsilon > 0$ and $\delta > 0$ such that

$$\epsilon < \hat{\epsilon} \quad \& \quad \delta < \hat{\epsilon} - u^{-1}\left(u(\epsilon)\frac{\varphi_j(T_j, (0, b)) + 0.5\varphi_j(T_j, \{b\})}{\varphi_j(T_j(0, b])}\right)$$

As $\epsilon \to 0$, it follows that such δ exists. Then, at $g_i(t_i) \in (b-\epsilon, b]$, we have

$$\begin{aligned} U_i(\psi_i, \psi_j, t_i) &\leq u(t_i - (b-\epsilon))(\varphi_j(T_j, (0, b)) + 0.5\varphi_j(T_j, \{b\})) \\ &< u(t_i - (b+\delta))(\varphi_j(T_j, (0, b)) + 0.5\varphi_j(T_j, \{b\})) \\ &< u(t_i - (b+\delta))(\varphi_j(T_j, (0, b)) + \varphi_j(T_j, \{b\})) \\ &= u(t_i - (b+\delta))(\varphi_j(T_j, (0, b])) \\ &= U_i(b+\delta, \psi_j, t_i) \end{aligned}$$

which is a contradiction. Hence, $\varphi_j(T_j, (b-\epsilon, b]) = 0$.

Step 3: We show by contradiction. Suppose $\varphi_i(T_i, \{b\}) > 0$ for some $b > 0$. Then, from step 1, we have $\varphi_j(T_j, (b-\epsilon, b]) > 0$ for every $\epsilon > 0$. Since

$\varphi_j(T_j,(b-\epsilon,b]) > 0$, from step 2, we have $\varphi_i(T_i,(b-\epsilon,b]) = 0$ and thus, $\varphi_i(T_i,\{b\}) = 0$, which is a contradiction. Hence, $\varphi_i(\{b\}) = 0$ for every $b > 0$.

We show $\Pr(i \text{ wins}|b_i)$ is continuous in every $b_i \in B_i$. Notice that

$$\Pr(i \text{ wins}|b_i) = \varphi_j(T_j,(0,b_i])$$

Since $\varphi_j(T_j,\{b_i\}) = 0$ for every b_i, it follows that $\varphi_j(T_j,(0,b_i])$ is continuous in every b_i. Hence, $\Pr(i \text{ wins}|b_i)$ is continuous in every $b_i \in B_i$. ■

Lemma A.7. *Suppose the profile of transition functions (ψ_i,ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i)$ is left continuous and $\Omega_i(t_i)$ is right continuous.*

Proof. We show $\Lambda_i(t_i) = \lim_{t_i'\uparrow t_i}\Omega_i(t_i')$. Since Ω_i is non-decreasing in t_i, we have

$$\lim_{t_i'\uparrow t_i}\Omega_i(t_i') = \sup_{t_i'\in(0,t_i)}\Omega_i(t_i')$$

From Lemma A.4, we know $\Omega_i(t_i') \le \Lambda_i(t_i)$ for every $t_i' < t_i$. Thus,

$$\lim_{t_i'\uparrow t_i}\Omega_i(t_i') \le \Lambda_i(t_i)$$

It remains to show $\lim_{t_i'\uparrow t_i}\Omega_i(t_i') \ge \Lambda_i(t_i)$. Consider a sequence $(t_{in})_{n=1}^{\infty}$ such that $t_{in}\uparrow t_i$ and $\lim_{n\to\infty}\Omega_i(t_{in}) = \sup_{t_i'<t_i}\Omega_i(t_i')$ for every $n \in \mathcal{N}$. Then,

$$U_i(\Omega_i(t_{in}),\psi_j,t_{in}) = u(t_{in}-\Omega_i(t_{in}))\Pr(i \text{ wins}|\Omega_i(t_{in}))$$

for every $n \in \mathcal{N}$. As $n\to\infty$, we get

$$U_i(\psi_i,\psi_j,t_i) = u(t_i - \sup_{t_i'\in(0,t_i)}\Omega_i(t_i'))\Pr(i \text{ wins}|\sup_{t_i'\in(0,t_i)}\Omega_i(t_i'))$$

Then,

$$\Lambda_i(t_i) \le \sup_{t_i'\in(0,t_i)}\Omega_i(t_i') = \lim_{t_i'\uparrow t_i}\Omega_i(t_i')$$

Since $\Lambda_i(t_i) \ge \sup_{t_i'\in(0,t_i)}\Omega_i(t_i')$ and $\Lambda_i(t_i) \le \sup_{t_i'\in(0,t_i)}\Omega_i(t_i')$, we have $\Lambda_i(t_i) = \sup_{t_i'<t_i}\Omega_i(t_i')$ for every $t_i' \in (0,a_i]$. Hence, $\Lambda_i(t_i) = \lim_{t_i'\uparrow t_i}\Omega_i(t_i')$. Therefore, Λ_i is left continuous. Similarly, it can be shown $\Omega_i(t_i) = \lim_{t_i'\downarrow t_i}\Lambda_i(t_i')$ for every $t_i' \in (0,a_i]$. ■

Since Λ_i is left continuous, $\Lambda_i(0) = \Omega_i(0)$ and $\Lambda_i(a_i) = \Omega_i(a_i)$, it follows if Λ_i is continuous, then $\Lambda_i(t_i) = \Omega_i(t_i)$ for every $t_i \in T_i$. Since Ω_i is right continuous, it follows if Ω_i is continuous, then $\Lambda_i(t_i) = \Omega_i(t_i)$ for every $t_i \in T_i$.

Lemma A.8. *Suppose the profile of transition functions (ψ_i,ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are strictly increasing in t_i for every $i \in N$.*

Proof. We show by contradiction. Suppose there exists $W_i \subseteq T_i$ such that Λ_i and Ω_i are non-increasing on W_i. From Lemma A.4, Λ_i and Ω_i are non-decreasing. Then, $\Lambda_i(t_i) = \Omega_i(t_i) = c$ for some $c > 0$ and for every $t_i \in W_i$. By Lemma A.6, $\varphi_i(T_i, (\Lambda_i(t_i)) = 0$, which is a contradiction. Therefore, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are strictly increasing in t_i for every $t_i \in T_i$ and every $i \in N$. ■

Lemma A.9. *Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. Then, $\Lambda_i(t_i)$ and $\Omega_i(t_i)$ are continuous for every $t_i \in T_i$ and $i \in N$.*

Proof. From Lemma A.7, Λ_i is left continuous. It remains to show Λ_i is right continuous. We show by contradiction. Suppose Λ_i is not right continuous at some $t_i \in T_i$. Then, for small enough $\epsilon > 0$, $\Pr(b_i \in (\Lambda_i(t_i), \Lambda_i(t_i + \epsilon)) = 0$. Thus $\Pr(b_j \in (\Lambda_i(t_i), \Lambda_i(t_i + \epsilon)) = 0$. Since $\Pr(b_j \in (\Lambda_i(t_i), \Lambda_i(t_i + \epsilon)) = 0$, it follows $U_i(\Lambda_i(t_i), \psi_j, t_i) > U_i(\Omega_i(t_i), \psi_j, t_i)$, which is a contradiction. Hence, Λ_i is right continuous at t_i. Therefore, $\Lambda_i(t_i)$ is continuous for every $t_i \in T_i$ and $i \in N$. As $\Lambda_i(t_i) = \Omega_i(t_i)$ if Λ_i is continuous, it follows that Ω_i is continuous in every $t_i \in T_i$ and $i \in N$. ■

B Appendix: Proofs

Proof of Theorem 1. Suppose the profile of transition functions (ψ_i, ψ_j) is a Bayesian equilibrium. From Lemma A.2 and Lemma A.9, it follows $\Lambda_i(t_i) = \Omega_i(t_i)$ for every $t_i \in T_i$. Thus, equilibrium strategy is pure, strictly increasing and continuous. (A) follows directly from Lemma A.2.

The optimization problem of bidder i is $\max_b F_j \circ \psi_j^{-1}(b)u(t_i - b)$. The first-order condition gives (2).

Conversely, suppose a profile of functions (ψ_i, ψ_j) satisfies (A) and (B). Consider bidder i with type t_i. The value function is

$$V_i(b, t_i) = F_j \circ \theta_j(b)u(t_i - b)$$

The first-order derivative is

$$\mathrm{D}_b V_i(b, t_i) = \mathrm{D}F_j \circ \theta_j(b)u(t_i - b) - F_j \circ \theta_j(b)u'(t_i - b)$$

Suppose bidder i over bids by bidding b' such that $\theta_i(b') > t_i$. Then,

$$\begin{aligned}\mathrm{D}_{b'} V_i(b', t_i) &= \mathrm{D}F_j \circ \theta_j(b')u(t_i - b') - F_j \circ \theta_j(b)u'(t_i - b') \\ &< \mathrm{D}F_j \circ \theta_j(b')u(\theta_i(b') - b') - F_j \circ \theta_j(b')u'(\theta_i(b') - b') \\ &= 0\end{aligned}$$

Hence, it is not profitable for bidder i to deviate. On the other hand, suppose bidder i under bids by bidding b'' such that $\theta_i(b'') < t_i$. Then,

$$\begin{aligned}\mathrm{D}_{b''} V_i(b'', t_i) &= \mathrm{D}F_j \circ \theta_j(b'')u(t_i - b) - F_j \circ \theta_j(b'')u'(t_i - b'') \\ &> \mathrm{D}F_j \circ \theta_j(b'')u(\theta_i(b'') - b'') - F_j \circ \theta_j(b'')u'(\theta_i(b'') - b'') \\ &= 0\end{aligned}$$

Hence, it is not profitable for bidder i to deviate. Therefore, (ψ_i, ψ_j) is an equilibrium profile. ■

Proof of Lemma 1. Consider (3). By the definition of Bayesian equilibrium, we have

$$u(t - \psi_i(t))F_j \circ \zeta_j(t) \geq u(t - \psi_j(t))F_j(t)$$

Interchanging the roles of i and j in the above equation, we have

$$u(t - \psi_j(t))F_i \circ \zeta_i(t) \geq u(t - \psi_i(t))F_i(t)$$

The above two equation implies

$$u(t - \psi_i(t))F_j \circ \zeta_j(t) > U_j(t) \geq u(t - \psi_i(t))F_i(t)$$

Thus, $F_j \circ \zeta_j(t) > F_i(t)$.

Since $F_j \circ \zeta_j(t) > F_i(t)$, it implies that there exists x such that $F_j \circ \zeta_j(t) = F_j(x) > F_i(t)$. From Theorem 1, such x is unique. Thus, $F_i \circ \zeta_i(x) = F_i(t) < F_j(x)$. ■

Proof of Lemma 2. We show (A) implies (B). Suppose $\psi_i(t) > \psi_j(t)$ for every $t \in (0, \bar{a})$. Then, $\zeta_i(t) < t$. Thus,

$$\mathcal{U}_i(t) = \frac{u(t - \psi_i(t))F_j \circ \zeta_j(t)}{u(t - \psi_j(t))F_i \circ \zeta_i(t)} \geq \frac{u(t - \psi_i(t))F_j(t)}{u(t - \psi_j(t))F_i \circ \zeta_i(t)} > \frac{F_j(t)}{F_i(t)} = \mathcal{G}_i(t)$$

Hence, $\mathcal{U}_i(t) > \mathcal{G}_i(t)$ for every $t \in (0, \bar{a})$.

We show (B) implies (A). Suppose $\mathcal{U}_i(t) > \mathcal{G}_i(t)$. Then,

$$u(t - \psi_i(t))F_j \circ \zeta_j(t)F_i(t) > u(t - \psi_j(t))F_i \circ \zeta_i(t)F_j(t)$$

Since $u(t - \psi_i(t))F_i(t) \leq u(t - \psi_j(t))F_j \circ \zeta_j(t)$, we have $F_j \circ \zeta_j(t) > F_j(t)$. Thus, $\psi_i(t) > \psi_j(t)$ for every $t \in (0, \bar{a})$.

We show (A) implies (C). Suppose $\psi_i(t) > \psi_j(t)$. Then,

$$\mathrm{D}\mathcal{U}_i = \frac{U_j \mathrm{D}U_i - U_i \mathrm{D}U_j}{U_j^2}$$

$\mathrm{D}\mathcal{U}_i > 0$ only if

$$\frac{U_j}{\mathrm{D}U_j} > \frac{U_i}{\mathrm{D}U_i}$$

or equivalently,

$$\frac{u(t - \psi_j)}{u'(t - \psi_j)} > \frac{u(t - \psi_i)}{u'(t - \psi_i)}$$

which is true as $u' > 0$, $u'' < 0$ and $\psi_i > \psi_j$.

It is straightforward to see (C) implies (A). ■

Proof of Theorem 5. Suppose $\theta_i(c) \leq \hat{\theta}_i(c)$ and $\theta_j(c) \leq \hat{\theta}_j(c)$ for every $c \in (0, \min\{\bar{b}, \hat{b}\})$. We show that there exists $\epsilon > 0$ such that

$$\theta_i(b) < \hat{\theta}_i(b) \quad \& \quad \frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \tag{P1}$$

for every $b \in [c - \epsilon, c)$.

Since $\theta_j(c) \leq \hat{\theta}_j(c)$, it follows that, for every $b < c$, we have $F_j \circ \theta_j(b) < F_j \circ \hat{\theta}_j(c)$ $\theta_i(b) < \hat{\theta}_i(b)$ and thus,

$$\frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(b)}$$

Case 1: $\theta_i(c) < \hat{\theta}_i(c)$ and $\theta_j(c) < \hat{\theta}_j(c)$

It is straightforward to see that there exists $\epsilon > 0$ such that (P1) holds for every $b \in [c - \epsilon, c)$.

Case 2: $\theta_i(c) = \hat{\theta}_i(c)$ and $\theta_j(c) < \hat{\theta}_j(c)$

From the system of differential equations given by (7) and (8), we have

$$\theta_i'(c) > \hat{\theta}_i'(c)$$

Then, there exists $\epsilon > 0$ such that (P1) holds for every $b \in [c - \epsilon, c)$.

Case 3: $\theta_i(c) < \hat{\theta}_i(c)$ and $\theta_j(c) = \hat{\theta}_j(c)$

It is straightforward to see that there exists $\epsilon > 0$ such that(P1) holds for every $b \in [c - \epsilon, c)$.

Case 4: $\theta_i(c) = \hat{\theta}_i(c)$ and $\theta_j(c) = \hat{\theta}_j(c)$

From the system of differential equations given by (7) and (8), we have

$$\theta_j'(c) > \hat{\theta}_j'(c)$$

Notice that

$$\mathrm{D} \log F_i \circ \theta_i(c) = \frac{u'(\theta_j(c) - c)}{u(\theta_j(c) - c)} \quad \& \quad \mathrm{D} \log F_i \circ \hat{\theta}_i(c) = \frac{u'(\hat{\theta}_j(c) - c)}{u(\hat{\theta}_j(c) - c)}$$

Differentiating the above equations w.r.t. c and using the fact that $\theta_i = \hat{\theta}_i$, $\theta_j = \hat{\theta}_j$ and $\theta_j' > \hat{\theta}_j'$, we get

$$\mathrm{D}^2 \log F_i \circ \theta_i(c) < \mathrm{D}^2 \log F_i \circ \hat{\theta}_i(c)$$

Thus, there exists $\epsilon > 0$ such that (P1) holds for every $b \in [c - \epsilon, c)$.

Let

$$M = \inf \left\{ x \in [0, c - \epsilon] \mid \theta_i(b) < \hat{\theta}_i(b) \quad \& \quad \frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \right.$$
$$\left. \text{for every } b \in (c - \epsilon, c) \right\}$$

We show that $M = 0$. We show by contradiction. Suppose $M > 0$. Then, either

$$\theta_i(M) = \hat{\theta}_i(M) \quad \text{or} \quad \frac{F_j \circ \theta_j(M)}{\hat{F}_j \circ \hat{\theta}_j(M)} = \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)}$$

Since $\theta_i(b) < \hat{\theta}_i(b)$ and $\theta_j(b) \leq \hat{\theta}_j(b)$ for every $b \in (M, c - \epsilon)$, it follows from (7) and (8) that

$$\frac{F_j \circ \theta_j(b)}{DF_j \circ \theta_j(b)} < \frac{\hat{F}_j \circ \hat{\theta}_j(b)}{D\hat{F}_j \circ \hat{\theta}_j(b)}$$

This implies

$$\mathrm{D} \log F_j \circ \theta_j(b) > \mathrm{D} \log \left\{ \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \hat{F}_j \circ \hat{\theta}_j(b) \right\}$$

Since $M < c - \epsilon$, we have

$$\log F_j \circ \theta_j(c - \epsilon) - \log F_j \circ \theta_j(M) > \log \left\{ \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \hat{F}_j \circ \hat{\theta}_j(c - \epsilon) \right\} - \log \left\{ \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \hat{F}_j \circ \hat{\theta}_j(M) \right\}$$

Rearranging the above terms, we get

$$\log \left\{ \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \hat{F}_j \circ \hat{\theta}_j(M) \right\} - \log F_j \circ \theta_j(M) > \log \left\{ \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \hat{F}_j \circ \hat{\theta}_j(c - \epsilon) \right\} - \log F_j \circ \theta_j(c - \epsilon)$$

From the definition of M, we have

$$\log \left\{ \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} \hat{F}_j \circ \hat{\theta}_j(c - \epsilon) \right\} - \log F_j \circ \theta_j(c - \epsilon) > 0$$

Thus,

$$\frac{F_j \circ \theta_j(M)}{\hat{F}_j \circ \hat{\theta}_j(M)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)}$$

Therefore, $\theta_i(M) = \hat{\theta}_i(M)$ must be the case. From the differential equations given by (7) and (8), the definition of M and the assumption of conditional stochastic dominance, we have

$$\frac{F_j \circ \theta_j(M)}{\hat{F}_j \circ \hat{\theta}_j(M)} < \frac{F_j \circ \hat{\theta}_j(c)}{\hat{F}_j \circ \hat{\theta}_j(c)} < \frac{F_j \circ \hat{\theta}_j(M)}{\hat{F}_j \circ \hat{\theta}_j(M)}$$

This implies $\theta_j(M) < \hat{\theta}_j(M)$. Since $\theta_i(M) = \hat{\theta}_i(M)$ and $\theta_j(M) < \hat{\theta}_j(M)$, we have $\theta_i'(M) > \hat{\theta}_i'(M)$. Thus, there exists $\delta > 0$ such that $\theta_i(M+\delta) > \hat{\theta}_i(M+\delta)$, which is a contradiction. Hence, $M = 0$.

Therefore, $\theta_i(b) < \hat{\theta}_i(b)$ and $F_j \circ \theta_i(b) < \pi(c)\hat{F}_j \circ \hat{\theta}_j(b)$ for every $b \in (0, c)$.

We show that $\theta_i(c) \leq \hat{\theta}_i(c)$ and $\theta_j(c) \leq \hat{\theta}_j(c)$ cannot hold simultaneously. We show by contradiction. Suppose there exists $c^* \in (0, \bar{b})$ such that $\theta_i(c^*) \leq \hat{\theta}_i(c^*)$ and $\theta_j(c^*) \leq \hat{\theta}_j(c^*)$. Then,

$$\frac{F_j \circ \theta_j(b)}{\hat{F}_j \circ \hat{\theta}_j(b)} < \frac{F_j \circ \hat{\theta}_j(c^*)}{\hat{F}_j \circ \hat{\theta}_j(c^*)}$$

Taking the limits $b \downarrow 0$, we have

$$\frac{F_j(0)}{\hat{F}_j(0)} < \frac{F_j \circ \hat{\theta}_j(c^*)}{\hat{F}_j \circ \hat{\theta}_j(c^*)}$$

which is a contradiction, as $F_j/\hat{F}_j$ is strictly increasing. Therefore, $\theta_i(c) \leq \hat{\theta}_i(c)$ and $\theta_j(c) \leq \hat{\theta}_j(c)$ cannot hold simultaneously.

We show that $\hat{b} > \bar{b}$. We show by contradiction. Suppose $\hat{b} \leq \bar{b}$. Then, $\theta_i(\hat{b}) \leq \hat{\theta}_i(\hat{b})$ and $\theta_j(\hat{b}) \leq \hat{\theta}_j(\hat{b})$, a contradiction. Hence, $\hat{b} > \bar{b}$.

We show $\theta_i(b) > \hat{\theta}_i(b)$ for every $b \in [0, \bar{b}]$. We show by contradiction. Since $\theta_i(\bar{b}) > \hat{\theta}_i(\bar{b})$, it follows that there exists $\epsilon > 0$ such that $\theta_i(b) > \hat{\theta}_i(b)$ for every $b \in (\bar{b} - \epsilon, \bar{b}]$. Suppose there exists b^* such that $\theta_i(b^*) = \hat{\theta}_i(b^*)$ and $\theta_i(b) > \hat{\theta}_i(b)$ for every $b \in (b^*, \bar{b}]$. Then, $\theta_j(b^*) > \hat{\theta}_j(b^*)$. From the differential equations given by (7) and (8), we have

$$\hat{\theta}_i'(b^*) > \theta_i'(b^*)$$

Thus, there exists $\delta > 0$ such that $\hat{\theta}_i(b^* + \delta) > \theta_i(b^* + \delta)$, which is a contradiction. Hence, $\theta_i(b) > \hat{\theta}_i(b)$ for every $b \in [0, \bar{b}]$.

We show $\hat{F}_j \circ \hat{\theta}_j(b) < F_j \circ \theta_j(b)$ for every $b \in (0, \bar{b})$. Since $\theta_i(b) > \hat{\theta}_i(b)$, from the differential equations given by (7) and (8), we have

$$\frac{F_j \circ \theta_j(b)}{\mathrm{D}F_j \circ \theta_j(b)} > \frac{\hat{F}_j \circ \hat{\theta}_j(b)}{\mathrm{D}\hat{F}_j \circ \hat{\theta}_j(b)}$$

Thus,

$$\mathrm{D}\left(\frac{\hat{F}_j \circ \hat{\theta}_j(b)}{F_j \circ \theta_j(b)}\right) > 0$$

Since $\hat{F}_j \circ \hat{\theta}_j(\bar{b}) < F_j \circ \theta_j(\bar{b}) = 1$, it follows $\hat{F}_j \circ \hat{\theta}_j(b) < F_j \circ \theta_j(b)$. ■

Proof of Corollary 5. When the distribution functions are (F_i, F_j), the equilibrium payoffs are

$$U_i(t_i, \psi_i, \psi_j) = \max_b F_j \circ \theta_j(b) u(t_i - b)$$
$$U_j(t_i, \psi_i, \psi_j) = \max_b F_i \circ \theta_i(b) u(t_j - b)$$

On the other hand, when the distribution functions are $(F_i, \hat{F}_j)$, the equilibrium payoffs are

$$\hat{U}_i(t_i, \hat{\psi}_i, \hat{\psi}_j) = \max_b \hat{F}_j \circ \hat{\theta}_j(b) u(t_i - b)$$
$$\hat{U}_j(t_i, \hat{\psi}_i, \hat{\psi}_j) = \max_b F_i \circ \hat{\theta}_i(b) u(t_j - b)$$

Since $\hat{F}_j \circ \hat{\theta}_j(b) < F_j \circ \theta_j(b)$ for every $b \in (0, \bar{b})$, it follows that

$$\hat{U}_j(t_j, \hat{\psi}_i, \hat{\psi}_j) < U_j(t_j, \psi_i, \psi_j)$$

Since $\hat{\theta}_i(b) < \theta_i(b)$ for every $b \in (0, \bar{b})$, it follows that

$$\hat{U}_i(t_i, \hat{\psi}_i, \hat{\psi}_j) < U_i(t_i, \psi_i, \psi_j)$$

■

Proof of Lemma 3. (A) Since $a_w = \theta_w(\bar{b}) < \theta_s(\bar{b}) = a_s$, it follows that there exists $\epsilon > 0$ such that $\theta_w(\bar{b} - \epsilon) < \theta_s(\bar{b} - \epsilon)$. Suppose there exists b^* such that $\theta_w(b^*) = \theta_s(b^*)$ and $\theta_w(b) < \theta_s(b)$ for every $b \in (b^*, \bar{b}]$. We show a contradiction. From (2) and assumption 1, we have

$$\begin{aligned}
\theta_s'(b^*) &= \frac{F_s \circ \theta_s(b^*)}{f_s \circ \theta_s(b^*)} \frac{u'(\theta_w(b^*) - b^*)}{u(\theta_w(b^*) - b^*)} \\
&< \frac{F_w \circ \theta_w(b^*)}{f_w \circ \theta_w(b^*)} \frac{u'(\theta_s(b^*) - b^*)}{u(\theta_s(b^*) - b^*)} \\
&= \theta_w'(b^*)
\end{aligned}$$

Then, there exists a $\delta > 0$ such that $\theta_S(b^* + \delta) < \theta_w(b^* + \delta)$, which is a contradiction. Hence, no such b^* exists. Therefore, $\theta_w(b) < \theta_s(b)$ for every $b \in (0, \bar{b}]$.

(B) Since $u' > 0$ and $u'' < 0$, we have $u(\theta_w(b) - b) < u(\theta_s(b) - b)$ and $u'(\theta_w(b) - b) > u'(\theta_s(b) - b)$. Since $\theta_w(b) < \theta_s(b)$, it follows from (2) that

$$\begin{aligned}
\frac{F_s \circ \theta_s(b)}{\mathrm{D}F_s \circ \theta_s(b)} &= \frac{u(\theta_w(b) - b)}{u'(\theta_w(b) - b)} \\
&< \frac{u(\theta_s(b) - b)}{u'(\theta_s(b) - b)} \\
&= \frac{F_w \circ \theta_w(b)}{\mathrm{D}F_w \circ \theta_w(b)}
\end{aligned}$$

This implies

$$\mathrm{D}\left(\frac{F_s \circ \theta_s(b)}{F_w \circ \theta_w(b)}\right) > 0$$

Since $F_s \circ \theta_s(\bar{b}) = F_w \circ \theta_w(\bar{b}) = 1$ and $\mathrm{D}(F_s \circ \theta_s(b)/F_w \circ \theta_w(b)) > 0$, it follows that $(F_s \circ \theta_s(b)/F_w \circ \theta_w(b)) < 1$. Therefore, $F_s \circ \theta_s(b) < F_w \circ \theta_w(b)$ for every $b \in (0, \bar{b})$. ■

Proof of Theorem 7. Let $G(b) = F_s \circ \theta_s(b) F_w \circ \theta_w(b)$ and $\tilde{G}(b) = \tilde{F}_s \circ \tilde{\theta}_s(b) \tilde{F}_w \circ \tilde{\theta}_w(b)$. To show $R(F_w, F_s) > R(\tilde{F}_w, \tilde{F}_s)$, it suffices to show $G(b) < \tilde{G}(b)$.

Notice that $G(0) = \tilde{G}(0) = 0$. We show there exists $\epsilon > 0$ such that $G(b) < \tilde{G}(b)$ for every $b \in (0, \epsilon)$. From the first-order condition given by (2), we have

$$\begin{aligned} \theta'_w(0) &= 1 + \frac{1}{c_s v'(0)}, \quad \theta'_s(0) = 1 + \frac{1}{c_w v'(0)} \\ \tilde{\theta}'_w(0) &= 1 + \frac{1}{\tilde{c}_s v'(0)}, \quad \tilde{\theta}'_s(0) = 1 + \frac{1}{\tilde{c}_w v'(0)} \end{aligned} \tag{17}$$

Consider $b \in \Re_{++}$. By mean value theorem, there exists $b^*_w \in (0, b)$ such that $\theta_w(b) = b\theta'_w(b^*_w)$. Similarly, $b^*_s, \tilde{b}^*_w$ and $\tilde{b}^*_s$ exists. So

$$\begin{aligned} \lim_{b \downarrow 0} \frac{\tilde{G}(b)}{G(b)} &= \lim_{b \downarrow 0} \frac{\tilde{\theta}_w(b)^{\tilde{c}_w} \tilde{\theta}_s(b)^{\tilde{c}_s}}{\theta_w(b)^{c_w} \theta_s(b)^{c_s}} \\ &= \lim_{b \downarrow 0} \frac{b^{\tilde{c}_w} \tilde{\theta}'_w(\tilde{b}^*_w)^{\tilde{c}_w} b^{\tilde{c}_s} \tilde{\theta}'_s(\tilde{b}^*_s)^{\tilde{c}_s}}{b^{c_w} \theta'_w(b^*_w)^{c_w} b^{c_s \theta'_s(b^*_s)^{c_s}}} \\ &= \frac{\tilde{\theta}'_w(0)^{\tilde{c}_w} \tilde{\theta}'_s(0)^{\tilde{c}_s}}{\theta'_w(0)^{c_w} \theta'_s(0)^{c_s}} \\ &= \frac{\left(1 + \frac{1}{\tilde{c}_s v'(0)}\right)^{\tilde{c}_w} \left(1 + \frac{1}{\tilde{c}_w v'(0)}\right)^{\tilde{c}_s}}{\left(1 + \frac{1}{c_s v'(0)}\right)^{c_w} \left(1 + \frac{1}{c_w v'(0)}\right)^{c_s}} \end{aligned}$$

We show

$$\begin{aligned} &\tilde{c}_w \log\left(1 + \frac{1}{\tilde{c}_s v'(0)}\right) + \tilde{c}_s \log\left(1 + \frac{1}{\tilde{c}_w v'(0)}\right) > \\ &c_w \log\left(1 + \frac{1}{c_s v'(0)}\right) + c_s \log\left(1 + \frac{1}{c_w v'(0)}\right) \end{aligned} \tag{18}$$

for every $c_w, c_s, \tilde{c}_w, \tilde{c}_s$ such that $c_w + c_s = \tilde{c}_w + \tilde{c}_s$ and $\tilde{c}_s > c_s \geq c_w > \tilde{c}_w$. Since $\tilde{c}_s > c_s \geq c_w > \tilde{c}_w$, there exists δ and δ' such that $c_w = \delta\tilde{c}_w + (1-\delta)\tilde{c}_s$ and $c_s = \delta'\tilde{c}_w + (1-\delta')\tilde{c}_s$. Define

$$k(x) = (c_w + c_s - x) \log\left(1 + \frac{1}{x v'(0)}\right)$$

for every $x < c_w + c_s$. Notice that k is a convex function and $\delta + \delta' = 1$. Then,

$$\begin{aligned} k(c_s) + k(c_w) &= k(\delta\tilde{c}_w + (1-\delta)\tilde{c}_s) + k(\delta'\tilde{c}_w + (1-\delta')\tilde{c}_s) \\ &< \delta k(\tilde{c}_w) + (1-\delta)k(\tilde{c}_s) + \delta' k(\tilde{c}_w) + (1-\delta')k(\tilde{c}_s) \\ &= k(\tilde{c}_w) + k(\tilde{c}_s) \end{aligned}$$

Hence, (18) holds. Therefore, there exists $\epsilon > 0$ such that $G(b) < \tilde{G}(b)$ for every $b \in (0, \epsilon)$.

To show $G(b) < \tilde{G}(b)$ for every $b \in \Re_{++}$, it suffices to show if $G(b) = \tilde{G}(b)$, then $g(b)/G(b) < \tilde{g}(b)/\tilde{G}(b)$.

We know

$$\frac{g(b)}{G(b)} = \frac{\mathrm{D}F_s \circ \theta_s(b)}{F_s \circ \theta_s(b)} + \frac{\mathrm{D}F_w \circ \theta_w(b)}{F_w \circ \theta_w(b)}$$

From (2), we have

$$\frac{g(b)}{G(b)} = \frac{u'(\theta_w(b) - b)}{u(\theta_w(b) - b)} + \frac{u'(\theta_s(b) - b)}{u(\theta_s(b) - b)} \tag{19}$$

Similarly,

$$\frac{\tilde{g}(b)}{\tilde{G}(b)} = \frac{u'(\tilde{\theta}_w(b) - b)}{u(\tilde{\theta}_w(b) - b)} + \frac{u'(\tilde{\theta}_s(b) - b)}{u(\tilde{\theta}_s(b) - b)} \tag{20}$$

Recall that $\theta_s(b) \geq \theta_w(b)$ and $\tilde{\theta}_s(b) > \tilde{\theta}_w(b)$. Six effective rankings are possible of (19) and (20) are

(E1) $\theta_s \geq \theta_w \geq \tilde{\theta}_s > \tilde{\theta}_w$

(E2) $\theta_s \geq \tilde{\theta}_s \geq \theta_w \geq \tilde{\theta}_w$ with at least one strict inequality

(E3) $\theta_s \geq \tilde{\theta}_s > \tilde{\theta}_w \geq \theta_w$

(E4) $\tilde{\theta}_s \geq \theta_s \geq \tilde{\theta}_w \geq \theta_w$ with at least one strict inequality

(E5) $\tilde{\theta}_s \geq \theta_s \geq \theta_w \geq \tilde{\theta}_w$ with at least one strict inequality

(E6) $\tilde{\theta}_s > \tilde{\theta}_w \geq \theta_s \geq \theta_w$

Suppose for some $b \in \Re_{++}$, $G(b) = \tilde{G}(b)$. Then, $\theta_s^{c_s}\theta_w^{c_w} = \tilde{\theta}_s^{\tilde{c}_s}\tilde{\theta}_w^{\tilde{c}_w}$. Let $z = c_s/(c_s + c_w)$ and $\tilde{z} = \tilde{c}_s/(\tilde{c}_s + \tilde{c}_w)$. Taking logarithms of the above equation, we have

$$z \log \theta_s(b) + (1 - z) \log \theta_w(b) = \tilde{z} \log \tilde{\theta}_s(b) + (1 - \tilde{z}) \log \tilde{\theta}_w(b)$$

Notice that $z < \tilde{z}$. This rules out (E1), (E4) and (E6).

We show (E3) cannot hold. We show if there exists some $b \in \Re_{++}$ such that (E3) holds, then for every $b' > b$,

$$\theta_s(b') > \tilde{\theta}_s(b') \geq \tilde{\theta}_w(b') > \theta_w(b') \tag{21}$$

We show a contradiction. Suppose (21) does not hold. Then, either

(i) $\theta_s(b') = \tilde{\theta}_s(b')$, $\mathrm{D}\theta_s(b') \leq \mathrm{D}\tilde{\theta}_s(b')$ & $\tilde{\theta}_w(b') \geq \theta_w(b')$, or

(ii) $\theta_w(b') = \tilde{\theta}_w(b')$, $\mathrm{D}\theta_w(b') \geq \mathrm{D}\tilde{\theta}_w(b')$& $\theta_s(b') \geq \tilde{\theta}_s(b')$

Suppose (i) is true. Then $\theta_s(b') = \tilde{\theta}_s(b')$ implies $c_s v(\theta_w(b') - b')\mathrm{D}\theta_s(b') = \tilde{c}_s v(\tilde{\theta}_w(b') - b')\mathrm{D}\tilde{\theta}_s(b')$. Since $\tilde{c}_s > c_s$ and $\mathrm{D}\theta_s(b') \leq \mathrm{D}\tilde{\theta}_s(b')$, it follows $\theta_w(b') > \tilde{\theta}_w(b')$, which is a contradiction.

Now suppose (ii) is true. Then $\theta_w(b') = \tilde{\theta}_w(b')$ implies $c_w v(\theta_s(b') - b')\mathrm{D}\theta_w(b') = \tilde{c}_w v(\tilde{\theta}_s(b') - b')\mathrm{D}\tilde{\theta}_w(b')$. Since $\tilde{c}_w < c_w$ and $\mathrm{D}\theta_w(b') \geq \mathrm{D}\tilde{\theta}_w(b')$, it follows $\theta_s(b') < \tilde{\theta}_s(b')$, which is a contradiction. Hence, (21) holds if (E3) holds. Since $\theta_s(\bar{b}) = \theta_w(\bar{b}) = \tilde{\theta}_s(\hat{b}) = \tilde{\theta}_w(\bar{b})$, it follows (E3) cannot hold.

It is straightforward to see (E2) implies $g(b)/G(b) < \tilde{g}(b)/\tilde{G}(b)$. We show (E5) implies $g(b)/G(b) < \tilde{g}(b)/\tilde{G}(b)$. Consider the following optimization problem

$$\max_{\{\theta_w,\theta_s,\tilde{\theta}_w,\tilde{\theta}_s\}} \frac{1}{v(\theta_w - b)} + \frac{1}{v(\theta_s - b)} - \frac{1}{v(\tilde{\theta}_w - b)} - \frac{1}{v(\tilde{\theta}_s - b)}$$

subject to

$$z \log \theta_s(b) + (1-z) \log \theta_w(b) = \tilde{z} \log \tilde{\theta}_s(b) + (1-\tilde{z}) \log \tilde{\theta}_w(b) \tag{C1}$$

$$\tilde{\theta}_s \geq \theta_s \tag{C2}$$

$$\theta_s \geq \theta_w \tag{C3}$$

The Lagrange of the optimization problem is

$$\begin{aligned}\mathcal{L}(\theta_w, \theta_s, \tilde{\theta}_w, \tilde{\theta}_s, l_1, l_2, l_3) = &\frac{1}{v(\theta_w - b)} + \frac{1}{v(\theta_s - b)} - \frac{1}{v(\tilde{\theta}_w - b)} - \frac{1}{v(\tilde{\theta}_s - b)} \\ &+ l_1\{z \log \theta_s(b) + (1-z) \log \theta_w(b) - \tilde{z} \log \tilde{\theta}_s(b) - (1-\tilde{z}) \log \tilde{\theta}_w(b)\} \\ &+ l_2(\tilde{\theta}_s - \theta_s) + l_3(\theta_s - \theta_w)\end{aligned}$$

The first-order conditions are

$$-\frac{v'(\theta_s - b)}{v(\theta_s - b)^2} + l_1 \frac{z}{\theta_s} - l_2 + l_3 = 0 \tag{i}$$

$$-\frac{v'(\theta_w - b)}{v(\theta_w - b)^2} + l_1 \frac{1-z}{\theta_w} - l_3 = 0 \tag{ii}$$

$$\frac{v'(\tilde{\theta}_s - b)}{v(\tilde{\theta}_s - b)^2} - l_1 \frac{\tilde{z}}{\tilde{\theta}_s} + l_2 = 0 \tag{iii}$$

$$\frac{v'(\tilde{\theta}_w - b)}{v(\tilde{\theta}_w - b)^2} - l_1 \frac{1-\tilde{z}}{\tilde{\theta}_w} = 0 \tag{iv}$$

$$\tilde{\theta}_s \geq \theta_s, \quad l_2 \geq 0 \text{ s.t. } (\tilde{\theta}_s - \theta_s) l_2 = 0 \tag{KT 1}$$

$$\theta_s \geq \theta_w, \quad l_3 \geq 0 \text{ s.t. } (\theta_s - \theta_w) l_3 = 0 \tag{KT 2}$$

Solving for l_1 from (i)-(iv), we have

$$\begin{aligned}l_1 &= \frac{\{v'(\theta_s - b) + (l_2 - l_3) v(\theta_s - b)^2\}\theta_s}{z v(\theta_s - b)^2} = \frac{\{v'(\theta_w - b) + l_3 v(\theta_w - b)^2\}\theta_w}{(1-z) v(\theta_w - b)^2} \\ &= \frac{\{v'(\tilde{\theta}_s - b) + l_2 v(\tilde{\theta}_s - b)^2\}\tilde{\theta}_s}{\tilde{z} v(\tilde{\theta}_s - b)^2} = \frac{v'(\tilde{\theta}_w - b)\tilde{\theta}_w}{(1-\tilde{z}) v(\tilde{\theta}_w - b)^2}\end{aligned}$$

Consider third and fourth equality. Since $\tilde{z} v(\tilde{\theta}_s - b)^2/\tilde{\theta}_s > (1-\tilde{z}) v(\tilde{\theta}_w - b)^2/\tilde{\theta}_w$, it follows $l_2 > 0$. From (KT 1), $\tilde{\theta}_s = \theta_s$. Consider second and fourth equality. Since $(1-z)v(\theta_w-b)^2/\theta_w > (1-\tilde{z})v(\tilde{\theta}_w-b)^2/\tilde{\theta}_w$, it follows $l_3>0$. From (KT 2), $\theta_s=\theta_w$. Hence, $\theta_s=\theta_w=\tilde{\theta}_s$. From (C1), $\theta_s=\theta_w=\tilde{\theta}_s=\tilde{\theta}_w$.

We check for the constraint qualification. The derivative matrix for constraints is

$$\begin{pmatrix} z/\theta_s & (1-z)/\theta_w & -\tilde{z}/\tilde{\theta}_s & -(1-\tilde{z})/\tilde{\theta}_w \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

The rank of the above matrix is 3. Hence constraint qualification is met. We show $\theta_s=\theta_w=\tilde{\theta}_s=\tilde{\theta}_w$ is a global maximum. Total differentiating (C1), we have

$$z\mathrm{d}\theta_s+(1-z)\mathrm{d}\theta_w=\tilde{z}\mathrm{d}\tilde{\theta}_s+(1-\tilde{z})\mathrm{d}\tilde{\theta}_w$$

This implies

$$\mathrm{d}\tilde{\theta}_w=\frac{z\mathrm{d}\theta_s+(1-z)\mathrm{d}\theta_w-\tilde{z}\tilde{\theta}_s}{1-\tilde{z}}$$

So,

$$\begin{aligned}
&\mathrm{d}\left\{\frac{1}{v(\theta_w-b)}+\frac{1}{v(\theta_s-b)}-\frac{1}{v(\tilde{\theta}_w-b)}-\frac{1}{v(\tilde{\theta}_s-b)}\right\}\\
&\quad=\frac{v'(\theta_s-b)}{v(\theta_s-b)^2}\{-\mathrm{d}\theta_s-\mathrm{d}\theta_w+\mathrm{d}\tilde{\theta}_s+\mathrm{d}\tilde{\theta}_w\}\\
&\quad=\frac{v'(\theta_s-b)}{v(\theta_s-b)^2}\left\{\frac{\tilde{z}+z-1}{1-\tilde{z}}\mathrm{d}\theta_s+\frac{\tilde{z}-z}{1-\tilde{z}}\mathrm{d}\theta_w-\frac{2\tilde{z}-1}{1-\tilde{z}}\mathrm{d}\tilde{\theta}_s\right\}
\end{aligned}$$

As $\tilde{z}>z\geq 1/2$, we have

$$\begin{aligned}
\frac{\tilde{z}+z-1}{1-\tilde{z}}\mathrm{d}\theta_s+\frac{\tilde{z}-z}{1-\tilde{z}}\mathrm{d}\theta_w-\frac{2\tilde{z}-1}{1-\tilde{z}}\mathrm{d}\tilde{\theta}_s &\leq \frac{z-\tilde{z}}{1-\tilde{z}}\mathrm{d}\theta_s+\frac{\tilde{z}-z}{1-\tilde{z}}\mathrm{d}\theta_w\\
&=\frac{z-\tilde{z}}{1-\tilde{z}}(\mathrm{d}\theta_s-\mathrm{d}\theta_w)\\
&\leq 0
\end{aligned}$$

Hence. $\theta_s=\theta_w=\tilde{\theta}_s=\tilde{\theta}_w$ is a global maximum. The value of the function at the optimum is 0. Therefore, the value function is strictly negative when (E5) holds.

So, at (E5) $g(b)/G(b)<\tilde{g}(b)/\tilde{G}(b)$. Hence, $G(b)<\tilde{G}(b)$ for every b. Therefore, $R(F_w,F_s)>R(\tilde{F}_w,\tilde{F}_s)$.

■

Proof of Theorem 8. We show that the ex-ante expected revenue of the seller is more in first-price auction format than in second-price auction. The expected revenue from second-price auction can be re-written as

$$R^{II}=ka_sS-k^{\tau_s+2}a_sJ$$

where

$$S = \frac{\tau_w + 1}{\tau_w + 2} \text{ and } J = \frac{\tau_w + 1}{(\tau_s + 2)(\tau_w + \tau_s + 3)}$$

It is easy to check that k is strictly increasing in τ_w. We need to show that Q is increasing in k. As k is increasing in τ_w, it suffices to show that Q is increasing in k. Notice that $k < 1$. Differentiating R^{II} w.r.t. k, we have

$$\begin{aligned} \mathrm{D}_k R^{II} &= a_s S - (\tau_s + 2)k^{\tau_s+1} J \\ &\geq a_s k^{\tau_s+1}(S - (\tau_s + 2)J) \\ &> 0 \end{aligned}$$

Therefore, R^{II} is strictly increasing in τ_w. Whenever $\tau_w \uparrow \tau_s$, k increases (and approaches 1) and thus R^{II} increases. Therefore,

$$\begin{aligned} R^{II} &= \frac{\tau_w + 1}{\tau_w + 2} k a_s - \frac{\tau_w + 1}{(\tau_s + 2)(\tau_w + \tau_s + 3)} k^{\tau_s+2} a_s \\ &< \frac{\tau_s + 1}{\tau_s + 2} a_s - \frac{\tau_s + 1}{(\tau_s + 2)(\tau_w + \tau_s + 3)} a_s \\ &= \frac{(\tau_s + 1)(\tau_w + \tau_s + 2)}{(\tau_s + 2)(\tau_w + \tau_s + 3)} a_s \\ &\leq \frac{(\tau_s + 1)(\tau_w + \tau_s + 2)}{(\tau_s + 1 + \alpha)(\tau_w + \tau_s + 3)} a_s \\ &= R^I \end{aligned}$$

Hence, the ex-ante expected revenue of the seller from first-price auction is more than that from the second-price auction. ■

References

[1] Krishna B Athreya and Soumendra N Lahiri. *Measure theory and probability theory*. Springer Science & Business Media, 2006.

[2] Estelle Cantillon. The effect of bidders' asymmetries on expected revenue in auctions. *Games and Economic Behavior*, 62(1):1–25, 2008.

[3] Harrison Cheng. Ranking sealed high-bid and open asymmetric auction. *Journal of Mathematical Economics*, 42:471–98, 2006.

[4] Jr Charles A. Holt. Competitive bidding for contracts under alternative auction procedures. *Journal of Political Economy*, 88(3):433–445, 1980.

[5] René Kirkegaard. Asymmetric first price auctions. *Journal of Economic Theory*, 144(4):1617–1635, 2009.

[6] Vijay Krishna. *Auction theory*. Academic press, 2009.

[7] Bernard Lebrun. Existence of an equilibrium in first price auctions. *Economic Theory*, 7(3):421–443, 1996.

[8] Bernard Lebrun. First price auctions in the asymmetric n bidder case. *Université Laval - Département d'économique (Working paper series),Cahiers de recherche*, (9715), 1997.

[9] Bernard Lebrun. Comparative statics in first price auctions. *Games and Economic Behavior*, 25(1):97–110, 1998.

[10] Bernard Lebrun. First price auctions in the asymmetric n bidder case. *International Economic Review*, 40(1):125–142, 1999.

[11] Huagang Li and John G Riley. Auction choice. *International Journal of Industrial Organization*, 25(6):1269–1298, 2007.

[12] Eric Maskin and John Riley. Optimal auctions with risk averse buyers. *Econometrica: Journal of the Econometric Society*, pages 1473–1518, 1984.

[13] Eric Maskin and John Riley. Asymmetric auctions. *The Review of Economic Studies*, 67(3):413–438, 2000.

[14] Eric Maskin and John Riley. Equilibrium in sealed high bid auctions. *The Review of Economic Studies*, 67(3):439–454, 2000.

[15] Eric Maskin and John Riley. Uniqueness of equilibrium in sealed high-bid auctions. *Games and Economic Behavior*, 45(2):395–409, 2003.

[16] Eric S Maskin and John G Riley. Auction theory with private values. *The American Economic Review*, 75(2):150–155, 1985.

[17] Steven Matthews. Comparing auctions for risk averse buyers: A buyer's point of view. *Econometrica: Journal of the Econometric Society*, pages 633–646, 1987.

[18] Roger B Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.

[19] M. Plum. Characterization and computation of nash equilibria for auctionswith incomplete information. *International Journal of Game Theory*, 20:393–418, 1992.

[20] John G Riley and William F Samuelson. Optimal auctions. *The American Economic Review*, 71(3):381–392, 1981.